Stat 351 Fall 2008
Assignment #9 Solutions

1. (a) By Definition I, we see that $X_1 - \rho X_2$ is normally distributed with mean

$$E(X_1 - \rho X_2) = E(X_1) - \rho E(X_2) = 0$$

and variance

$$\operatorname{var}(X_1 - \rho X_2) = \operatorname{var}(X_1) + \rho^2 \operatorname{var}(X_2) - 2\rho \operatorname{cov}(X_1, X_2) = 1 + \rho^2 - 2\rho^2 = 1 - \rho^2.$$

That is, $X_1 - \rho X_2 = Y$ where $Y \in N(0, 1-\rho^2)$. Hence, $Y = \sqrt{1-\rho^2} Z$ where $Z \in N(0,1)$. In other words, there exists a $Z \in N(0,1)$ such that

$$X_1 - \rho X_2 = \sqrt{1-\rho^2} Z.$$

1. (b) Since $\mathbf{X} = (X_1, X_2)'$ is MVN, and since

$$Z = \frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}},$$

we conclude that $(Z, X_2)'$ is also a MVN. Hence, we know from Theorem V.7.1 that the components of a MVN are independent if and only if they are uncorrelated. We find

$$\begin{aligned}\operatorname{cov}(Z, X_2) = \operatorname{cov}\left(\frac{X_1}{\sqrt{1-\rho^2}} - \frac{\rho X_2}{\sqrt{1-\rho^2}}, X_2\right) &= \frac{1}{\sqrt{1-\rho^2}} \operatorname{cov}(X_1, X_2) - \frac{\rho}{\sqrt{1-\rho^2}} \operatorname{var}(X_2) \\ &= \frac{\rho}{\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \\ &= 0\end{aligned}$$

which verifies that Z and X_2 are, in fact, independent.

Exercise 5.3, page 129: Let

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem V.3.1, $\mathbf{Y}$ is MVN with mean

$$B\overline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7/2 & 1/2 & -1 \\ 1/2 & 1/2 & 0 \\ -1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

Hence, we see that $\mathbf{Y} \in \mathcal{N}(\overline{0}, \mathbf{\Sigma})$ where

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix}.$$

We now compute $\det[\mathbf{\Sigma}] = 10 - 9 = 1$ and

$$\mathbf{\Sigma}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 2 \end{pmatrix}.$$

If we write $\mathbf{y} = (y_1, y_2, y_3)'$, then

$$\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} = y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2$$

and so the density of $\mathbf{Y}$ is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{1}{2}(y_1^2 + 5y_2^2 - 6y_2y_3 + 2y_3^2)\right\}.$$

Note that this problem could also be solved by observing that $Y_1 \in \mathcal{N}(0,1)$ and

$$(Y_2, Y_3)' \in \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right)$$

are independent so that $f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) \cdot f_{Y_2,Y_3}(y_2, y_3)$.

Problem #27, page 147: In order to determine the values of a and b for which $\mathbb{E}(U - a - bV)^2$ is a minimum, we must minimize the function $g(a,b) = \mathbb{E}(U - a - bV)^2$. If $U = X_1 + X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$, then

$$U - a - bV = X_1 + X_2 + X_3 - a - b(X_1 + 2X_2 + 3X_3) = (1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a.$$

Notice that $\mathbb{E}(U - a - bV)^2 = \operatorname{var}(U - a - bV) + [\mathbb{E}(U - a - bV)]^2$. We now compute

$$\begin{aligned} \operatorname{var}(U - a - bV) &= \operatorname{var}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a) \\ &= (1-b)^2 \operatorname{var}(X_1) + (1-2b)^2 \operatorname{var}(X_2) + (1-3b)^2 \operatorname{var}(X_3) \\ &= (1-b)^2 + (1-2b)^2 + (1-3b)^2 \end{aligned}$$

using the fact that X_1, X_2, X_3 are i.i.d. $\mathcal{N}(1,1)$. Furthermore,

$$\begin{aligned} \mathbb{E}(U - a - bV) = \mathbb{E}((1-b)X_1 + (1-2b)X_2 + (1-3b)X_3 - a) &= (1-b) + (1-2b) + (1-3b) - a \\ &= 3 - 6b - a \end{aligned}$$

which implies that

$$g(a,b) = (1-b)^2 + (1-2b)^2 + (1-3b)^2 + [3 - 6b - a]^2 = 12 - 48b + 50b^2 - 6a + 12ab + a^2.$$

To minimize g, we begin by finding the critical points. That is,

$$\frac{\partial}{\partial a} g(a,b) = -6 + 12b + 2a = 0$$

implies $a + 6b = 3$, and

$$\frac{\partial}{\partial b} g(a,b) = -48 + 100b + 12a = 0$$

implies $25b + 3a = 12$. Solving the second equation for b yields

$$25b = 12 - 3a = 12 - 3(3 - 6b) \quad \text{and so} \quad b = \frac{3}{7}.$$

Substituting in gives

$$a = 3 - 6b = 3 - \frac{18}{7} = \frac{3}{7}.$$

Since

$$\frac{\partial^2}{\partial a^2} g(a,b) = 2 > 0$$

and

$$\frac{\partial^2}{\partial a^2} g(a,b) \cdot \frac{\partial^2}{\partial b^2} g(a,b) - \left(\frac{\partial^2}{\partial a \partial b} g(a,b)\right)^2 = 2 \cdot 100 - 12^2 = 56 > 0$$

we conclude by the second derivative test that $a = 3/7$, $b = 3/7$ is indeed the minimum.

3. From Midterm #2, we find that the orthogonal decompostion of $\mathbf{\Lambda}$ is given by $\mathbf{\Lambda} = CDC'$ where

$$C = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, if we let $\mathbf{Y} = C'\mathbf{X}$, then $\mathbf{Y} \in \mathcal{N}(\mathbf{0}, D)$ so that

$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = \mathbf{y}'D^{-1}\mathbf{y} = Q(\mathbf{y}).$$

Setting $Q(\mathbf{y}) = 1$ gives

$$\frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2} = y_1^2 + \frac{y_2^2}{2} = 1.$$

This describes an ellipse centred at the origin passing through the points $(0, \sqrt{2})'$, $(0, -\sqrt{2})'$, $(1, 0)'$, and $(-1, 0)'$. We now notice that we can write C as

$$C = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}.$$

This matrix describes a counterclockwise rotation by $\pi/3 = 60°$. Thus,

$$Q(\mathbf{x}) = \mathbf{x}'\mathbf{\Lambda}^{-1}\mathbf{x} = \frac{5}{8}x_1^2 + \frac{\sqrt{3}}{4}x_1x_2 + \frac{7}{8}x_2^2 = 1$$

describes the same ellipse rotated by $\pi/3$. In other words, it is an ellipse passing through the points

$$C\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} = \left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)', \quad C\begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} = \left(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)',$$

$$C\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)', \quad C\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)'.$$

4. (a) If $X \in U(0,1)$, then the distribution function of X is

$$F_X(x) = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \ge 1. \end{cases}$$

Therefore, if $Y = -\log X$, then the distribution function of Y is

$$F_Y(y) = P\{Y \le y\} = P\{-\log X \le y\} = P\{X \ge e^{-y}\} = 1 - P\{X \le e^{-y}\} = 1 - F_X(e^{-y}) = 1 - e^{-y}$$

provided that $y > 0$. We recognize this as the distribution function of an Exp(1) random variable. That is, $Y \in \text{Exp}(1)$.

4. (b) If $Y_i = -\log X_i$ for $i = 1, \dots, n$, then by part **(a)**, we know that $Y_1, \dots, Y_n$ are i.i.d. Exp(1) random variables. Furthermore,

$$\prod_{i=1}^n X_i = \exp\left\{\log \prod_{i=1}^n X_i\right\} = \exp\left\{\sum_{i=1}^n \log X_i\right\} = \exp\left\{-\sum_{i=1}^n Y_i\right\}.$$

If we now let

$$Z = \sum_{i=1}^n Y_i,$$

then using moment generating functions it follows that $Z \in \Gamma(n,1)$ (or it follows from Problem #20 in Chapter I). Finally, if we let

$$W = \prod_{i=1}^n X_i = e^{-Z},$$

then the distribution function of W is

$$F_W(w) = P\{W \le w\} = P\{e^{-Z} \le w\} = P\{Z \ge -\log w\} = 1 - P\{Z \le -\log w\} = 1 - F_Z(-\log w).$$

Hence, the density function of W is

$$f_W(w) = \frac{1}{w} f_Z(-\log w) = \frac{1}{w} \cdot \frac{1}{\Gamma(n)} (-\log w)^{n-1} e^{\log w} = \frac{(-\log w)^{n-1}}{\Gamma(n)}, \quad 0 < w < 1.$$

5. (a) Since $S_{n+1} = S_n + Y_{n+1}$ we see that

$$\text{cov}(S_n, S_{n+1}) = \text{cov}(S_n, S_n + Y_{n+1}) = \text{cov}(S_n, S_n) + \text{cov}(S_n, Y_{n+1}) = \text{var}(S_n) + 0$$

using the fact that Y_{n+1} is independent of S_n. Furthermore, since $S_n = Y_1 + \cdots + Y_n$ and $Y_1, Y_2, \dots, Y_n$ are independent, we find

$$\text{var}(S_n) = \text{var}(Y_1 + \cdots + Y_n) = \text{var}(Y_1) + \text{var}(Y_2) + \cdots + \text{var}(Y_n) = 1 + 1 + \cdots + 1 = n.$$

Thus, we conclude $\text{cov}(S_n, S_{n+1}) = n$.

5. (b) Without loss of generality, suppose that $x \ge 0$. In order for the simple random walk to be at position $2x$ at time $2n$, it must be the case that $2x$ steps "to the right" were taken to reach position $2x$, and then of the remaining $2n - 2x$ steps, $n - x$ were taken "to the right" and $n - x$

were taken "to the left." In other words, there must have been $2x + n - x = n + x$ steps "to the right" and $n - x$ steps "to the left." There are $\binom{2n}{n+x}$ ways to do this, so that the probability is given by

$$P(S_{2n} = 2x) = \binom{2n}{n+x} 2^{-2n}, \quad x = -n, -n+1, \ldots, n-1, n.$$

(Note that, by symmetry, the probability is the same if x is replaced by $-x$.)

6. (a) The probability that fewer than two calls come in the first hour is $P(T_2 > 1)$. However, $\{T_2 > 1\} = \{X_1 < 2\}$ so it is equivalent to calculate either $P(T_2 > 1)$ or $P(X_1 < 2)$. Since $X_1 \in \text{Po}(4)$, we find

$$P(X_1 < 2) = P(X_1 = 0) + P(X_1 = 1) = \frac{4^0}{0!}e^{-4} + \frac{4^1}{1!}e^{-4} = 5e^{-4}.$$

On the other hand, since $T_2 \in \Gamma(2, \frac{1}{4})$, we compute

$$P(T_2 > 1) = \int_1^\infty \frac{1}{\Gamma(2)} 4^2 x e^{-4x}\,dx = \int_1^\infty 16xe^{-4x}\,dx = 5e^{-4}.$$

Note that integration by parts with $u = x$ and $dv = 16e^{-4x}$ gives

$$\int 16xe^{-4x}\,dx = -4xe^{-4x} + \int 4e^{-4x}\,dx = -4xe^{-4x} - e^{-4x}.$$

6. (b) The probability that at least two calls arrive in the second hour given that six calls arrive in the first hour is

$$\begin{aligned} P(X_2 \geq 8 | X_1 = 6) &= \frac{P(X_2 \geq 8, X_1 = 6)}{P(X_1 = 6)} = \frac{P(X_2 - X_1 \geq 2, X_1 = 6)}{P(X_1 = 6)} \\ &= \frac{P(X_2 - X_1 \geq 2) P(X_1 = 6)}{P(X_1 = 6)} = P(X_2 - X_1 \geq 2). \end{aligned}$$

Since $X_2 - X_1 \in \text{Po}(4)$, we conclude that

$$P(X_2 - X_1 \geq 2) = 1 - P(X_2 - X_1 < 2) = 1 - 5e^{-4}$$

using our result in **6. (a)**.

6. (c) Note that T_{15} is the time that the fifteenth call arrives. Since $T_{15} \in \Gamma(15, \frac{1}{4})$, we conclude

$$E(T_{15}) = 15 \cdot \frac{1}{4} = \frac{15}{4}.$$

Alternatively, since $T_{15} = \tau_1 + \tau_2 + \cdots + \tau_{15}$ with $\tau_i \in \text{Exp}(\frac{1}{4})$, we conclude

$$E(T_{15}) = E(\tau_1) + E(\tau_2) + \cdots + E(\tau_{15}) = \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4} = \frac{15}{4}.$$

6. (d) The probability that exactly 5 calls arrive in the first hour given that eight calls arrive in the first two hours is given by

$$\begin{aligned} P(X_1 = 5 | X_2 = 8) = \frac{P(X_1 = 5, X_2 = 8)}{P(X_2 = 8)} &= \frac{P(X_2 - X_1 = 3, X_1 = 5)}{P(X_2 = 8)} \\ &= \frac{P(X_2 - X_1 = 3) P(X_1 = 5)}{P(X_2 = 8)}. \end{aligned}$$

Since $X_2 - X_1 \in \text{Po}(4)$,

$$P(X_2 - X_1 = 3) = \frac{4^3}{3!}e^{-4},$$

since $X_1 \in \text{Po}(4)$,

$$P(X_1 = 5) = \frac{4^5}{5!}e^{-4},$$

and since $X_2 \in \text{Po}(8)$,

$$P(X_2 = 8) = \frac{8^8}{8!}e^{-8},$$

we can combine everything to conclude

$$P(X_1 = 5 | X_2 = 8) = \frac{\frac{4^3}{3!}e^{-4}\frac{4^5}{5!}e^{-4}}{\frac{8^8}{8!}e^{-8}} = \frac{8!}{3!\,5!\,2^8} = \frac{7}{32}.$$