Stat 351 Fall 2007
Assignment #7 Solutions

1. If X_1, X_2 are independent $N(0,1)$ random variables, then by Definition I, $Y_1 = X_1 + 3X_2 - 2$ is normal with mean $E(Y_1) = E(X_1) + 3E(X_2) - 2 = -2$ and variance $\text{var}(Y_1) = \text{var}(X_1 + 3X_2 - 2) = \text{var}(X_1) + 9\,\text{var}(X_2) + 6\,\text{cov}(X_1, X_2) = 1 + 9 + 0 = 10$, and $Y_2 = X_1 - 2X_2 + 1$ is normal with mean $E(Y_2) = E(X_1) - 2E(X_2) + 1 = 1$ and variance $\text{var}(Y_2) = \text{var}(X_1 - 2X_2 + 1) = \text{var}(X_1) + 4\,\text{var}(X_2) - 4\,\text{cov}(X_1, X_2) = 1 + 4 - 0 = 5$. Since $\text{cov}(Y_1, Y_2) = \text{cov}(X_1 + 3X_2 - 2, X_1 - 2X_2 + 1) = \text{var}(X_1) + \text{cov}(X_1, X_2) - 6\,\text{var}(X_2) = 1 + 0 - 6 = -5$, we conclude that $\mathbf{Y} = (Y_1, Y_2)'$ is multivariate normal $N(\boldsymbol{\mu}, \mathbf{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{pmatrix} 10 & -5 \\ -5 & 5 \end{pmatrix}.$$

2. Let

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & -2 \\ 3 & -2 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 9 & 36 \end{pmatrix}.$$

3. Let

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 0 \\ 1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -1 \\ 4 & 9 & -2 \\ -1 & -2 & 13 \end{pmatrix}.$$

4. Let

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$$

so that $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and covariance matrix

$$B\mathbf{\Lambda}B' = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 5 & 1 \\ -2 & 1 & 5 \end{pmatrix}.$$

5 (a). Since the conditional distribution of $Y|X = x \in N(x, x^2)$, we know that $\mathbb{E}(Y|X) = X$ and $\text{var}(Y|X) = X^2$. Therefore, it follows from Theorem II.2.1 that

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \frac{1}{2}$$

and from Corollary II.2.3.1 that

$$\text{var}(Y) = \mathbb{E}(\text{var}(Y|X)) + \text{var}(\mathbb{E}(Y|X)) = \mathbb{E}(X^2) + \text{var}(X) = \frac{1}{12} + \left(\frac{1}{2}\right)^2 + \frac{1}{12} = \frac{5}{12}.$$

By definition, $\text{cov}(X, Y) = \mathbb{E}(XY) + \mathbb{E}(Y)\mathbb{E}(X)$. In order to calculate $\mathbb{E}(XY)$, notice that Theorem II.2.1 implies $\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|X))$. However, by Theorem II.2.2 ("taking out what is known"), we see that

$$\mathbb{E}(\mathbb{E}(XY|X)) = \mathbb{E}(X\mathbb{E}(Y|X)) = \mathbb{E}(X \cdot X) = \mathbb{E}(X^2) = \frac{1}{12} + \left(\frac{1}{2}\right)^2.$$

Combining everything gives

$$\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(Y)\mathbb{E}(X) = \frac{1}{12} + \left(\frac{1}{2}\right)^2 - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

5 (b). We begin by noticing that $f_X(x) = 1$, $0 < x < 1$, and

$$f_{Y|X=x}(y) = \frac{1}{x\sqrt{2\pi}} \exp\left\{-\frac{1}{2x^2}(y - x)^2\right\}, \quad -\infty < y < \infty.$$

This implies that the joint density of X and Y is

$$f_{X,Y}(x, y) = f_{Y|X=x}(y) \cdot f_X(x) = \frac{1}{x\sqrt{2\pi}} \exp\left\{-\frac{1}{2x^2}(y - x)^2\right\}$$

provided $0 < x < 1$ and $-\infty < y < \infty$. Now let $U = Y/X$ and $V = X$ so that solving for X and Y gives

$$X = V \quad \text{and} \quad Y = UV.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ v & u \end{vmatrix} = -v.$$

The density of (U, V) is therefore given by

$$\begin{aligned} f_{U,V}(u, v) = f_{X,Y}(v, uv) \cdot |J| = v f_{X,Y}(v, uv) &= v \cdot \frac{1}{v\sqrt{2\pi}} \exp\left\{-\frac{1}{2v^2}(uv - v)^2\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(u - 1)^2\right\} \end{aligned}$$

provided that $0 < v < 1$, and $-\infty < u < \infty$. We see that we can write $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$ where $f_V(v) = 1$, $0 < v < 1$, and

$$f_U(u) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(u-1)^2\right\}, \quad -\infty < u < \infty.$$

Hence, we conclude that $X \in U(0,1)$ and $Y/X \in N(1,1)$ are independent.

5 (c). Notice that we can write

$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{Y}{X} \cdot X\right).$$

Since Y/X and X are independent we conclude

$$\mathbb{E}\left(\frac{Y}{X} \cdot X\right) = \mathbb{E}\left(\frac{Y}{X}\right)\mathbb{E}(X).$$

Hence,

$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{Y}{X}\right)\mathbb{E}(X)$$

or, in other words,

$$\mathbb{E}\left(\frac{Y}{X}\right) = \frac{\mathbb{E}(Y)}{\mathbb{E}(X)}.$$

6. By definition,

$$\operatorname{corr}(X^2, Y^2) = \frac{\operatorname{cov}(X^2, Y^2)}{\sqrt{\operatorname{var}(X^2)\operatorname{var}(Y^2)}}$$

where

$$\operatorname{cov}(X^2, Y^2) = \mathbb{E}(X^2Y^2) - \mathbb{E}(X^2)\mathbb{E}(Y^2).$$

We also note that if

$$\mathbf{X} = (X, Y)' \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right),$$

then $X \in N(0,1)$, $Y \in N(0,1)$, and $\operatorname{cov}(X,Y) = \operatorname{corr}(X,Y) = \rho$. Suppose that $Z \in N(0,1)$ so that the moment generating function of Z is

$$\psi_Z(t) = e^{t^2/2}.$$

Computing the fourth derivative gives

$$\psi_Z^{(4)}(t) = (t^4 + 6t^2 + 3)e^{t^2/2}$$

and so we conclude that

$$\mathbb{E}(Z^4) = \psi_Z^{(4)}(0) = 3.$$

Hence, we conclude that

$$\operatorname{var}(X^2) = \mathbb{E}(X^4) - [\mathbb{E}(X^2)]^2 = 3 - 1 = 2 \quad \text{and} \quad \operatorname{var}(Y^2) = \mathbb{E}(Y^4) - [\mathbb{E}(Y^2)]^2 = 3 - 1 = 2$$

since $\mathbb{E}(X^2) = \operatorname{var}(X) = 1$ and $\mathbb{E}(Y^2) = \operatorname{var}(Y) = 1$. The last thing we need to compute is $\mathbb{E}(X^2Y^2)$. Notice that Theorems II.2.1 and II.2.2 imply that

$$\mathbb{E}(X^2Y^2) = \mathbb{E}(\mathbb{E}(X^2Y^2|X)) = \mathbb{E}(X^2\mathbb{E}(Y^2|X)).$$

However, we see that $\mathbb{E}(Y^2|X) = \text{var}(Y|X) + [\mathbb{E}(Y|X)]^2$ and so we can finish the calculation if we can determine the conditional distribution of $Y|X = x$. Fortunately, this calculation is done for us in Section V.6. That is, it is shown that $Y|X = x \in N(\rho x, 1 - \rho^2)$. Hence,

$$\mathbb{E}(Y^2|X) = \text{var}(Y|X) + [\mathbb{E}(Y|X)]^2 = 1 - \rho^2 + (\rho X)^2 = 1 - \rho^2 + X^2\rho^2$$

and so

$$\begin{aligned}\mathbb{E}(X^2Y^2) = \mathbb{E}(X^2\mathbb{E}(Y^2|X)) = \mathbb{E}(X^2(1 - \rho^2 + X^2\rho^2)) &= \mathbb{E}(X^2) - \rho^2\mathbb{E}(X^2) + \rho^2\mathbb{E}(X^4) \\ &= 1 - \rho^2 + 3\rho^2 \\ &= 1 + 2\rho^2\end{aligned}$$

using our earlier facts that $\mathbb{E}(X^2) = 1$ and $\mathbb{E}(X^4) = 3$. Finally, we have all the pieces to conclude that

$$\text{corr}(X^2, Y^2) = \frac{\text{cov}(X^2, Y^2)}{\sqrt{\text{var}(X^2)\,\text{var}(Y^2)}} = \frac{\mathbb{E}(X^2Y^2) - \mathbb{E}(X^2)\mathbb{E}(Y^2)}{\sqrt{\text{var}(X^2)\,\text{var}(Y^2)}} = \frac{1 + 2\rho^2 - 1}{\sqrt{2 \cdot 2}} = \rho^2$$

as required.