Stat 351 Fall 2007
Assignment #8 Solutions

Exercise 7.1, page 134: By Definition I, we know that X and $Y - \rho X$ are normally distributed. Therefore, by Theorem 7.1, X and $Y - \rho X$ are independent if and only if $\operatorname{cov}(X, Y - \rho X) = 0$. We compute

$$\begin{aligned}\operatorname{cov}(X, Y-\rho X) = \operatorname{cov}(X,Y) - \operatorname{cov}(X, \rho X) = \operatorname{cov}(X,Y) - \rho \operatorname{var}(X) &= \rho \operatorname{SD}(X)\operatorname{SD}(Y) - \rho \operatorname{var}(X) \\ &= \rho \operatorname{var}(X) - \rho \operatorname{var}(X) = 0\end{aligned}$$

since $\operatorname{SD}(X)\cdot\operatorname{SD}(Y) = \operatorname{SD}(X)\cdot\operatorname{SD}(X) = \operatorname{var}(X)$ by the assumption that $\operatorname{var}(X) = \operatorname{var}(Y)$. Hence, X and $Y - \rho X$ are in fact independent.

Problem #9, page 144: Note that by Theorem 7.1, in order to show X_1, X_2, and X_3 are independent, it is enough to show that $\operatorname{cov}(X_1,X_2) = \operatorname{cov}(X_1,X_3) = \operatorname{cov}(X_2,X_3) = 0$. Thus, if X_1 and $X_2 + X_3$ are independent, then $\operatorname{cov}(X_1, X_2+X_3) = \operatorname{cov}(X_1,X_2) + \operatorname{cov}(X_1,X_3) = 0$ and so

$$\operatorname{cov}(X_1,X_2) = -\operatorname{cov}(X_1,X_3). \tag{1}$$

If X_2 and $X_1 + X_3$ are independent, then $\operatorname{cov}(X_2, X_1+X_3) = \operatorname{cov}(X_2,X_1) + \operatorname{cov}(X_2,X_3) = 0$ and so

$$\operatorname{cov}(X_2,X_1) = -\operatorname{cov}(X_2,X_3). \tag{2}$$

Finally, if X_3 and X_1+X_2 are independent, then $\operatorname{cov}(X_3, X_1+X_2) = \operatorname{cov}(X_3,X_1)+\operatorname{cov}(X_3,X_2) = 0$ and so

$$\operatorname{cov}(X_3,X_1) = -\operatorname{cov}(X_3,X_2). \tag{3}$$

Since (1), (2), and (3) must be simultaneously satisfied, the only possibility is that $\operatorname{cov}(X_1,X_2) = \operatorname{cov}(X_1,X_3) = \operatorname{cov}(X_2,X_3) = 0$. Hence, X_1, X_2, and X_3 are independent as required.

Problem #10, page 145: From Assignment #7, we know that the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}10 & 5\\ 5 & 5\end{pmatrix}\right)$$

and so we see that $Y_1 \in N(2,10)$, $Y_2 \in N(-1,5)$, and $\operatorname{corr}(Y_1,Y_2) = \frac{1}{\sqrt{2}}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = y$ is normal with mean $2 + \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{10}}{\sqrt{5}}(y-(-1)) = y+3$ and variance $10\left(1-\left(\frac{1}{\sqrt{2}}\right)^2\right) = 5$. That is,

$$Y_1|Y_2 = y \in N(y+3, 5).$$

Problem #11, page 145: From Assignment #7, we know that the distribution of $\mathbf{Y} = (Y_1, Y_2)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix}0\\8\end{pmatrix}, \begin{pmatrix}16 & -2\\ -2 & 16\end{pmatrix}\right)$$

and so we see that $Y_1 \in N(0,16)$, $Y_2 \in N(8,16)$, and $\operatorname{corr}(Y_1,Y_2) = -\frac{1}{8}$. Thus, by the results in Section V.6, the distribution of $Y_1|Y_2 = 10$ is normal with mean $0 - \frac{1}{8}\cdot\frac{4}{4}(10-8) = -\frac{1}{4}$ and variance $16\left(1-\left(-\frac{1}{8}\right)^2\right) = \frac{63}{4}$. That is,

$$Y_1|Y_2 = 10 \in N\left(-\tfrac{1}{4}, \tfrac{63}{4}\right).$$

Problem #13, page 145: From Assignment #7, we know that the distribution of $\mathbf{X} = (X_1, X_2, X_3)'$ is

$$\mathbf{X} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2 & 4 & -5\\4 & 9 & -10\\-5 & -10 & 13\end{pmatrix}\right)$$

and so we see that $X_1 \in N(0,2)$, $X_2 \in N(0,9)$, and $X_3 \in N(0,13)$. Since $\text{cov}(X_1, X_3) = -5$, we conclude that $X_1 + X_3 \in N(0,5)$. Finally, we compute $\text{cov}(X_2, X_1 + X_3) = \text{cov}(X_2, X_1) + \text{cov}(X_2, X_3) = 4 - 10 = -6$ and so $\text{corr}(X_2, X_1 + X_3) = -\frac{2}{\sqrt{5}}$. Thus, by the results in Section V.6, the distribution of $X_2 | X_1 + X_3 = x$ is normal with mean $0 - \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{5}}(x - 0) = -\frac{6x}{5}$ and variance $9\left(1 - \left(-\frac{2}{\sqrt{5}}\right)^2\right) = \frac{9}{5}$. That is,

$$X_2 | X_1 + X_3 = x \in N\left(-\tfrac{6x}{5}, \tfrac{9}{5}\right).$$

Problem #14, page 145: From Assignment #7, we know that the distribution of $\mathbf{Y} = (Y_1, Y_2, Y_3)'$ is

$$\mathbf{Y} \in N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{pmatrix}\right).$$

By definition,

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{f_{Y_1,Y_2,Y_3}(y,0,0)}{f_{Y_2,Y_3}(0,0)}.$$

From Definition III, we know

$$f_{Y_1,Y_2,Y_3}(y,0,0) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}$$

since

$$\begin{pmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix}3 & -1 & -1\\-1 & 3 & -1\\-1 & -1 & 3\end{pmatrix}.$$

The joint distribution of $(Y_2, Y_3)'$ is

$$(Y_2, Y_3)' \in N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\right)$$

and so

$$f_{Y_2,Y_3}(0,0) = \frac{1}{2\pi\sqrt{3}}.$$

Thus, we conclude

$$f_{Y_1|Y_2=0,Y_3=0}(y) = \frac{\left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{\sqrt{4}}\, e^{-\frac{1}{2}\frac{3}{4}y^2}}{\frac{1}{2\pi\sqrt{3}}} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{3}}{2} \exp\left\{-\frac{1}{2}\left(\frac{y}{2/\sqrt{3}}\right)^2\right\}$$

which we recognize as the density function of a normal random variable with mean 0 and variance $3/4$. That is,

$$Y_1 | Y_2 = Y_3 = 0 \in N\left(0, \tfrac{3}{4}\right).$$

Problem #3, page 143: If the random vector $(X, Y)'$ has a multivariate normal distribution, then it follows from Definition I that both $X + Y$ and $X - Y$ are normal random variables. If $\text{var}(X) = \text{var}(Y)$, then

$$\text{cov}(X + Y, X - Y) = \text{cov}(X, X) - \text{cov}(X, Y) + \text{cov}(Y, X) + \text{cov}(Y, Y) = \text{var}(X) - \text{var}(Y) = 0.$$

Theorem V.7.1 therefore implies that $X+Y$ and $X-Y$ are independent since $\text{cov}(X+Y, X-Y) = 0$.

Problem #12, page 145: Let $\mathbf{X} = (X_1, X_2, X_3)'$ where X_1, X_2, X_3 are i.i.d. $N(1, 1)$ so that $\mathbf{X} \in N(\boldsymbol{\mu}, \boldsymbol{\Lambda})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{Y} = (U, V)'$ where $U = 2X_1 - X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$. If

$$B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

then $\mathbf{Y} = B\mathbf{X}$. By Theorem 3.1, $\mathbf{Y}$ is MVN with mean

$$B\boldsymbol{\mu} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

and covariance matrix

$$B\boldsymbol{\Lambda}B' = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 14 \end{pmatrix}.$$

We can immediately conclude that $U \in N(2, 6)$, $V \in N(6, 14)$, and $\text{cov}(U, V) = 3$ so that $\text{corr}(U, V) = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}}$. It follows from Section V.6 that the distribution of $V|U = u$ is

$$N\left(6 + \frac{3}{2\sqrt{21}} \frac{\sqrt{14}}{\sqrt{6}}(u - 2), 14\left(1 - \frac{9}{4 \cdot 21}\right)\right).$$

Choosing $u = 3$ therefore implies that

$$V|U = 3 \in N\left(6.5, 12.5\right).$$