Stat 351 Fall 2007
Solutions to Assignment #5

1. (a) We begin by calculating $\mathbb{E}(Y_1)$. That is,

$$\mathbb{E}(Y_1) = 1 \cdot P(Y = 1) + (-1) \cdot P(Y = -1) = p - (1-p) = 2p - 1.$$

We now notice that $S_{n+1} = S_n + Y_{n+1}$. Therefore,

$$\begin{aligned}
\mathbb{E}(S_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}(S_n + Y_{n+1}|X_1, \ldots, X_n) \\
&= \mathbb{E}(S_n|X_1, \ldots, X_n) + \mathbb{E}(Y_{n+1}|X_1, \ldots, X_n) \\
&= S_n + \mathbb{E}(Y_{n+1}) \\
&= S_n + 2p - 1
\end{aligned}$$

and so

$$\begin{aligned}
\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}(S_{n+1} - (n+1)(2p-1)|X_1, \ldots, X_n) \\
&= \mathbb{E}(S_{n+1}|X_1, \ldots, X_n) - (n+1)(2p-1) \\
&= S_n + 2p - 1 - (n+1)(2p-1) \\
&= S_n - n(2p-1) \\
&= X_n.
\end{aligned}$$

We can now conclude that $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

1. (b) Notice that

$$Z_{n+1} = \left(\frac{1-p}{p}\right)^{S_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n + Y_{n+1}} = \left(\frac{1-p}{p}\right)^{S_n} \left(\frac{1-p}{p}\right)^{Y_{n+1}}$$

Therefore,

$$\begin{aligned}
\mathbb{E}(Z_{n+1}|X_1, \ldots, X_n) &= \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{S_n} \left(\frac{1-p}{p}\right)^{Y_{n+1}} |X_1, \ldots, X_n\right) \\
&= \left(\frac{1-p}{p}\right)^{S_n} \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}} |X_1, \ldots, X_n\right) \\
&= \left(\frac{1-p}{p}\right)^{S_n} \mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right)
\end{aligned}$$

We now compute

$$\mathbb{E}\left(\left(\frac{1-p}{p}\right)^{Y_{n+1}}\right) = p\left(\frac{1-p}{p}\right)^{1} + (1-p)\left(\frac{1-p}{p}\right)^{-1} = (1-p) + p = 1$$

and so we conclude

$$\mathbb{E}(Z_{n+1}|X_1, \ldots, X_n) = \left(\frac{1-p}{p}\right)^{S_n} = Z_n.$$

Hence, $\{Z_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.

2. We interpret the conditional probabilities given in the problem to mean

$$P(X_{n+1} = (1-q)x_n | X_n = x_n) = 1 - x_n \quad \text{and} \quad P(X_{n+1} = q + (1-q)x_n | X_n = x_n) = x_n.$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(X_{n+1} | X_1 = x_1, \ldots, X_n = x_n) \\
&\quad = (1-q)x_n \cdot P(X_{n+1} = (1-q)x_n | X_n = x_n) + (q + (1-q)x_n) \cdot P(X_{n+1} = q + (1-q)x_n | X_n = x_n) \\
&\quad = (1-q)x_n \cdot (1 - x_n) + (q + (1-q)x_n) \cdot x_n \\
&\quad = (1-q)x_n - (1-q)x_n^2 + qx_n + (1-q)x_n^2 \\
&\quad = x_n - qx_n + qx_n \\
&\quad = x_n.
\end{aligned}$$

In other words,

$$\mathbb{E}(X_{n+1} | X_1, \ldots, X_n) = X_n$$

and so we conclude that $\{X_n, n = 1, 2, \ldots\}$ is, in fact, a martingale.