Stat 351 Fall 2007
Assignment #2 Solutions

Problem 2: We verify that $Q(A)$ is a probability by checking the three conditions.

- Since $P(\emptyset) = 0$, we conclude

  $$Q(\emptyset) = P(\emptyset|B) = \frac{P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$$

  since $\{\emptyset \cap B\} = \emptyset$. Similarly,

  $$Q(\Omega) = P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

  since $\{\Omega \cap B\} = B$.

- We observe that $A^c \cup A = \Omega$ so that

  $$B = B \cap \Omega = (A^c \cup A) \cap B = (A^c \cap B) \cup (A \cap B).$$

  Since $(A^c \cap B)$ and $(A \cap B)$ are disjoint, we conclude that

  $$P(B) = P((A^c \cap B) \cup (A \cap B)) = P(A^c \cap B) + P(A \cap B).$$

  Dividing both sides by $P(B)$ gives

  $$\frac{P(B)}{P(B)} = \frac{P(A^c \cap B)}{P(B)} + \frac{P(A \cap B)}{P(B)}.$$

  In other words, $1 = Q(A^c) + Q(A)$, or $Q(A^c) = 1 - Q(A)$, as required.

- If $A_1, A_2, \ldots$ are disjoint, then since $(A_1 \cup A_2 \cup \cdots) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots$ with $(A_i \cap B) \cap (A_j \cap B) = \emptyset$ for $i \neq j$, we conclude that

  $$P((A_1 \cup A_2 \cup \cdots) \cap B) = P((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots$$

  and so

  $$\begin{aligned} Q(A_1 \cup A_2 \cup \cdots) = \frac{P((A_1 \cup A_2 \cup \cdots) \cap B)}{P(B)} &= \frac{P(A_1 \cap B) + P(A_2 \cap B) + \cdots}{P(B)} \\ &= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \cdots \\ &= Q(A_1) + Q(A_2) + \cdots. \end{aligned}$$

Problem 3 (Exercise 1.2): This exercise was discussed in class; we just complete the missing details. Since $f_{X,Y}(x, y) = 1/\pi$ for $x^2 + y^2 \leq 1$, we have

$$\mathbb{E}(XY) = \iint_{x^2+y^2 \leq 1} xy \cdot \frac{1}{\pi} \cdot dx\, dy.$$

To compute this double integral, we use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le 1$, $0 \le \theta < 2\pi$, $dx\,dy = r\,dr\,d\theta$, and so

$$\begin{aligned}
\mathbb{E}(XY) = \iint_{x^2+y^2\le 1} xy \cdot \frac{1}{\pi} \cdot dx\,dy &= \int_0^{2\pi}\int_0^1 r\cos\theta \cdot r\sin\theta \cdot \frac{1}{\pi} \cdot r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{r^3}{\pi}\cos\theta\sin\theta\,dr\,d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \cos\theta\sin\theta\,d\theta \\
&= \frac{1}{8\pi}\int_0^{2\pi} \sin(2\theta)\,d\theta \\
&= \frac{1}{16\pi}\cos(2\theta)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Furthermore, we find

$$\mathbb{E}(X) = \int_{-1}^1 x \cdot \frac{2}{\pi}\sqrt{1-x^2}\,dx \quad \text{and} \quad \mathbb{E}(Y) = \int_{-1}^1 y \cdot \frac{2}{\pi}\sqrt{1-y^2}\,dy.$$

Therefore, since both of these integrals are the same, we only need to evaluate one of them. Thus, letting $u = 1 - x^2$ so that $du = -2x\,dx$, we find

$$\mathbb{E}(Y) = \mathbb{E}(X) = \int_{-1}^1 x \cdot \frac{2}{\pi}\sqrt{1-x^2}\,dx = -\frac{1}{\pi}\int_0^0 \sqrt{u}\,du = 0.$$

Hence, we conclude that $\text{cov}(X,Y) = \mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y) = 0$ and so X and Y are, in fact, dependent but uncorrelated random variables.

Problem 4 (Exercise 1.3): If (X,Y) is uniformly distributed on the square with corners $(\pm 1, \pm 1)$, then the joint density of (X,Y) is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}, & \text{if } -1 \le x \le 1, -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

- The marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

  If $-1 \le x \le 1$, then the range of possible y values is $-1 \le y \le 1$, and so

$$f_X(x) = \int_{-1}^1 \frac{1}{4}\,dy = \frac{1}{2}.$$

  That is,

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, if $-1 \le y \le 1$, then the range of possible x values is $-1 \le x \le 1$, and so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-1}^{1} \frac{1}{4}\,dx = \frac{1}{2}.$$

That is,

$$f_Y(y) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$, we conclude that X and Y are independent.

- If X and Y are independent, then they are necessarily uncorrelated since $E(XY) = E(X)E(Y)$ so that

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0.$$

Problem 5 (Exercise 1.1): Since the volume of the unit sphere in $\mathbb{R}^3$ is $4\pi/3$, the joint density of (X,Y,Z) is

$$f_{X,Y,Z}(x,y,z) = \begin{cases} \frac{3}{4\pi}, & \text{if } x^2+y^2+z^2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

- Therefore, the marginal density of (X,Y) is given by

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz.$$

If x, y, z are constrained to have $x^2+y^2+z^2 \le 1$, then for fixed x with $-1 \le x \le 1$, the range of possible y values is $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, and that the range of z is $-\sqrt{1-x-y^2} \le z \le \sqrt{1-x^2-y^2}$. It therefore follows that

$$f_{X,Y}(x,y) = \int_{-\sqrt{1-x-y^2}}^{\sqrt{1-x^2-y^2}} \frac{3}{4\pi}\,dz = \frac{3}{2\pi}\sqrt{1-x^2-y^2}$$

for $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. In other words,

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2\pi}\sqrt{1-x^2-y^2}, & \text{if } x^2+y^2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

- The marginal density of X is then given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz\,dy.$$

From our work above, we find that if $-1 \le x \le 1$, then

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy.$$

This can be solved with a u-substitution. Let $y = \left(\sqrt{1-x^2}\right) \cdot \sin u$ so that

$$dy = \left(\sqrt{1-x^2}\right) \cdot \cos u\,du$$

and so

$$\begin{aligned}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy &= \frac{3}{2\pi}(1-x^2)\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \left(\sqrt{1-\sin^2 u}\right)\cdot \cos u\,du \\ &= \frac{3}{2\pi}(1-x^2)\int_{-\pi/2}^{\pi/2} \cos^2 u\,du.\end{aligned}$$

being careful to watch our new limits of integration and remembering that $\sin^{-1}(-1) = -\pi/2$ and $\sin^{-1}(1) = \pi/2$. Recalling the half-angle identities for cosine, we find

$$\int \cos^2 u\,du = \int \frac{1}{2} + \frac{1}{2}\cos(2u)\,du = \frac{u}{2} + \frac{1}{4}\sin(2u)$$

and so

$$\begin{aligned}\frac{3}{2\pi}(1-x^2)\int_{-\pi/2}^{\pi/2} \cos^2 u\,du &= \frac{3}{2\pi}(1-x^2)\left[\frac{u}{2} + \frac{1}{4}\sin(2u)\right]_{-\pi/2}^{\pi/2} \\ &= \frac{3}{2\pi}(1-x^2)\left[\frac{\pi/2}{2} - \frac{-\pi/2}{2}\right] \\ &= \frac{3}{4}(1-x^2).\end{aligned}$$

In summary,

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2), & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise.}\end{cases}$$

Note: You can check that f_X is, in fact, a density by verifying that

$$\int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx = 1.$$