Statistics 351 Fall 2006 (Kozdron) Midterm #1 — Solutions

1. (a) By definition, $f_{X|Y=y}(x)$ is given by

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

We being by calculating

$$f_Y(y) = \int_0^{\sqrt{4-y^2}} \frac{1}{\pi}\, dx = \frac{1}{\pi}\sqrt{4-y^2}$$

for $0 < y < 2$. Therefore,

$$f_{X|Y=y}(x) = \frac{\frac{1}{\pi}}{\frac{1}{\pi}\sqrt{4-y^2}} = \frac{1}{\sqrt{4-y^2}}, \quad 0 < x < \sqrt{4-y^2}.$$

1. (b) We find

$$E(X|Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y=y}(x)\, dx = \int_0^{\sqrt{4-y^2}} x \cdot \frac{1}{\sqrt{4-y^2}}\, dx = \frac{1}{2} \frac{\left(\sqrt{4-y^2}\right)^2}{\sqrt{4-y^2}} = \frac{\sqrt{4-y^2}}{2}.$$

1. (c) Solving for X and Y we find

$$X = U\cos V \quad \text{and} \quad Y = U \sin V.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u \sin^2 v = u.$$

The joint density of (U,V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u\cos v, u \sin v) \cdot |J| = \frac{u}{\pi}$$

provided that $0 < u < 2$ and $0 < v < \frac{\pi}{2}$.

1. (d) From **(c)**, we see that U and V are independent since we can write $f_{U,V}(u,v) = f_U(u) \cdot f_V(v)$ where $f_U(u) = \frac{u}{2}$, $0 < u < 2$, and $f_V(v) = \frac{2}{\pi}$, $0 < v < \frac{\pi}{2}$.

2. By the law of total probability,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|M=m}(x) f_M(m)\, dm.$$

Since $X|M = m \in U(0,m)$, we know that $f_{X|M=m}(x) = \frac{1}{m}$, $0 < x < m$. We also find that

$$f_M(m) = F_M'(m) = 12m^2 - 12m^3 = 12m^2(1-m), \quad 0 < m < 1.$$

Therefore,

$$f_X(x) = \int_x^1 \frac{1}{m} \cdot 12m^2(1-m)\, dm = (6m^2 - 4m^3)\Big|_x^1 = 2 - 6x^2 + 4x^3, \quad 0 < x < 1.$$

3. Notice that $P(X_{(1)} = X_1) = P(X_1 < X_2)$. Therefore, by the law of total probability,

$$P(X_1 < X_2) = \int_0^\infty P(X_2 > x | X_1 = x) \cdot f_{X_1}(x)\, dx.$$

Since X_1 and X_2 are independent, we find

$$P(X_2 > x | X_1 = x) = P(X_2 > x) = \int_x^\infty 2e^{-2y}\, dy = e^{-2x}.$$

Thus,

$$P(X_1 < X_2) = \int_0^\infty e^{-2x} \cdot e^{-x}\, dx = -\frac{1}{3} e^{-3x} \Big|_0^\infty = \frac{1}{3}.$$

An alternative solution can be given by conditioning on the value of X_2 instead. By the law of total probability,

$$P(X_1 < X_2) = \int_0^\infty P(X_1 < x | X_2 = x) \cdot f_{X_2}(x)\, dx.$$

Since X_1 and X_2 are independent, we find

$$P(X_1 < x | X_2 = x) = P(X_1 < x) = \int_0^x e^{-y}\, dy = 1 - e^{-x}.$$

Thus,

$$P(X_1 < X_2) = \int_0^\infty (1 - e^{-x}) \cdot 2e^{-2x}\, dx = 2\left[-\frac{1}{2}e^{-2x} + \frac{1}{3}e^{-3x}\right]_0^\infty = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

4. (a) Notice that we can write $Z = Y \cdot X_3$, and that X_3 is independent of Y. Therefore,

$$\begin{aligned} E(Z|Y) &= E(Y \cdot X_3 | Y) \\ &= Y E(X_3|Y) \quad \text{by "taking out what is known"} \\ &= Y E(X_3) \quad \text{since } X_3 \text{ is independent of } Y \\ &= 3Y \end{aligned}$$

4. (b) Recalling that $\text{cov}(X_1, X_2) = E(X_1 \cdot X_2) - E(X_1) \cdot E(X_2)$, we find

$$\begin{aligned} E(Z) = E(E(Z|Y)) = E(3Y) = 3E(Y) &= 3E(X_1 \cdot X_2) \\ &= 3\left[\text{cov}(X_1, X_2) + E(X_1) \cdot E(X_2)\right] \\ &= 3[6 + 1 \cdot 2] \\ &= 24. \end{aligned}$$