Stat 351 Fall 2006
Assignment #9 Solutions

(a) The probability that fewer than two calls come in the first hour is $P(T_2 > 1)$. However, $\{T_2 > 1\} = \{X_1 < 2\}$ so it is equivalent to calculate either $P(T_2 > 1)$ or $P(X_1 < 2)$. Since $X_1 \in \text{Po}(4)$, we find

$$P(X_1 < 2) = P(X_1 = 0) + P(X_1 = 1) = \frac{4^0}{0!}e^{-4} + \frac{4^1}{1!}e^{-4} = 5e^{-4}.$$

On the other hand, since $T_2 \in \Gamma(2, \frac{1}{4})$, we compute

$$P(T_2 > 1) = \int_1^\infty \frac{1}{\Gamma(2)} 4^2 x e^{-4x}\, dx = \int_1^\infty 16 x e^{-4x}\, dx = 5e^{-4}.$$

Note that integration by parts with $u = x$ and $dv = 16e^{-4x}$ gives

$$\int 16xe^{-4x}\, dx = -4xe^{-4x} + \int 4e^{-4x}\, dx = -4xe^{-4x} - e^{-4x}.$$

(b) The probability that at least two calls arrive in the second hour given that six calls arrive in the first hour is

$$\begin{aligned} P(X_2 \geq 8 | X_1 = 6) &= \frac{P(X_2 \geq 8, X_1 = 6)}{P(X_1 = 6)} = \frac{P(X_2 - X_1 \geq 2, X_1 = 6)}{P(X_1 = 6)} \\ &= \frac{P(X_2 - X_1 \geq 2) P(X_1 = 6)}{P(X_1 = 6)} = P(X_2 - X_1 \geq 2). \end{aligned}$$

Since $X_2 - X_1 \in \text{Po}(4)$, we conclude that

$$P(X_2 - X_1 \geq 2) = 1 - P(X_2 - X_1 < 2) = 1 - 5e^{-4}$$

using our result in **(a)**.

(c) Note that T_{15} is the time that the fifteenth call arrives. Since $T_{15} \in \Gamma(15, \frac{1}{4})$, we conclude

$$E(T_{15}) = 15 \cdot \frac{1}{4} = \frac{15}{4}.$$

Alternatively, since $T_{15} = \tau_1 + \tau_2 + \cdots + \tau_{15}$ with $\tau_i \in \text{Exp}(\frac{1}{4})$, we conclude

$$E(T_{15}) = E(\tau_1) + E(\tau_2) + \cdots + E(\tau_{15}) = \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{4} = \frac{15}{4}.$$

(d) The probability that exactly 5 calls arrive in the first hour given that eight calls arrive in the first two hours is given by

$$\begin{aligned} P(X_1 = 5 | X_2 = 8) = \frac{P(X_1 = 5, X_2 = 8)}{P(X_2 = 8)} &= \frac{P(X_2 - X_1 = 3, X_1 = 5)}{P(X_2 = 8)} \\ &= \frac{P(X_2 - X_1 = 3) P(X_1 = 5)}{P(X_2 = 8)}. \end{aligned}$$

Since $X_2 - X_1 \in \text{Po}(4)$,

$$P(X_2 - X_1 = 3) = \frac{4^3}{3!}e^{-4},$$

since $X_1 \in \text{Po}(4)$,

$$P(X_1 = 5) = \frac{4^5}{5!}e^{-4},$$

and since $X_2 \in \text{Po}(8)$,

$$P(X_2 = 8) = \frac{8^8}{8!}e^{-8},$$

we can combine everything to conclude

$$P(X_1 = 5 | X_2 = 8) = \frac{\frac{4^3}{3!}e^{-4}\frac{4^5}{5!}e^{-4}}{\frac{8^8}{8!}e^{-8}} = \frac{8!}{3!\,5!\,2^8} = \frac{7}{32}.$$