Stat 351 Fall 2006
Assignment #6 Solutions

Problem #3, page 115: If $0 \le y \le 1/2$, then

$$f_Y(y) = \int_y^{1-y} f_{X_{(1)},X_{(2)}}(y,z)\,dz = \int_y^{1-y} 2\,dz = 2(1-2y).$$

On the other hand, if $1/2 \le y \le 1$, then

$$f_Y(y) = \int_{1-y}^{y} f_{X_{(1)},X_{(2)}}(z,1-y)\,dz = \int_{1-y}^{y} 2\,dz = 2(2y-1).$$

Problem #6, page 115: Since $E[F(X_{(n)}) - F(X_{(1)})] = E[F(X_{(n)})] - E[F(X_{(1)})]$, we compute each of $E[F(X_{(n)})]$ and $E[F(X_{(1)})]$ separately. Therefore, by definition,

$$E[F(X_{(n)})] = \int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n.$$

From Theorem IV.1.2, we know that $f_{X_{(n)}}(y_n) = n[F_{X_{(n)}}(y_n)]^{n-1} f(y_n)$ so that

$$\int_{-\infty}^{\infty} F(y_n) f_{X_{(n)}}(y_n)\,dy_n = \int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,dy_n.$$

Making the substitution $u = F(y_n)$ so that $du = F'(y_n)dy_n = f(y_n)dy_n$ gives

$$\int_{-\infty}^{\infty} n[F(y_n)]^n f(y_n)\,dy_n = \int_0^1 nu^n\,du = \frac{n}{n+1}.$$

Note that since F is a distribution, our new limits of integration are $F(-\infty) = 0$ and $F(\infty) = 1$. As for $E[F(X_{(1)})]$, using Theorem IV.1.2, we compute

$$E[F(X_{(1)})] = \int_{-\infty}^{\infty} F(y_1) f_{X_{(1)}}(y_1)\,dy_1 = \int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,dy_1.$$

Making the same substitution as above gives

$$\int_{-\infty}^{\infty} F(y_1) n[1-F(y_1)]^{n-1} f(y_1)\,dy_1 = \int_0^1 nu(1-u)^{n-1}\,du = n\int_0^1 (1-v)v^{n-1}\,dv = 1 - \frac{n}{n+1}.$$

Finally, we combine our two results to conclude that

$$E[F(X_{(n)}) - F(X_{(1)})] = \frac{n}{n+1} - \left[1 - \frac{n}{n+1}\right] = \frac{n-1}{n+1}.$$

Problem #9, page 116: **(a)**: If X_1 and X_2 are independent Exp(a) random variables, then by Theorem IV.2.1, the joint density of $(X_{(1)}, X_{(2)})$ is given by

$$f_{X_{(1)},X_{(2)}}(y_1,y_2) = \begin{cases} \frac{2}{a^2}\exp\left(-\frac{y_1+y_2}{a}\right), & \text{for } 0 < y_1 < y_2 < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $U = X_{(1)}$ and let $V = X_{(2)} - X_{(1)}$. Solving for $X_{(1)}$ and $X_{(2)}$ gives

$$X_{(1)} = U \quad \text{and} \quad X_{(2)} = U + V.$$

The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

Therefore, by Theorem I.2.1, the density of (U, V) is given by

$$f_{U,V}(u,v) = f_{X_{(1)},X_{(2)}}(u, u+v)\cdot|J| = \frac{2}{a^2}\exp\left(-\frac{u+u+v}{a}\right) = \frac{2}{a^2}\exp\left(-\frac{2u+v}{a}\right) = \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}$$

provided that $v > 0$ and $u > 0$. The marginal density of U is

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_0^{\infty} \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}\,dv = \frac{2}{a}e^{-2u/a}$$

for $u > 0$. We recognize that this is the density of an exponential random variable with parameter $a/2$; that is, $U = X_{(1)} \in \text{Exp}(a/2)$. The marginal density of V is

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_0^{\infty} \frac{2}{a}e^{-2u/a}\cdot\frac{1}{a}e^{-v/a}\,du = \frac{1}{a}e^{-v/a}$$

for $v > 0$. We recognize that this is the density of an exponential random variable with parameter a; that is, $V = X_{(2)} - X_{(1)} \in \text{Exp}(a)$. Since we can express $f_{U,V}(u,v) = f_U(u)\cdot f_V(v)$ we conclude that U and V are independent; in other words, $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent.

(b): To compute $E(X_{(2)}|X_{(1)} = y)$, we can use properties of conditional expectation (Theorem II.2.2):

$$\begin{aligned} E(X_{(2)}|X_{(1)} = y) &= E(X_{(2)} - X_{(1)} + X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}|X_{(1)} = y) + E(X_{(1)}|X_{(1)} = y) \\ &= E(X_{(2)} - X_{(1)}) + y \\ &= a + y \end{aligned}$$

where the first expression after the third equality follows since $X_{(2)} - X_{(1)}$ is independent of $X_{(1)}$ and the second expression follows since $X_{(1)}$ is "known" when conditioned on the value $X_{(1)} = y$.

As for $E(X_{(1)}|X_{(2)} = x)$, we need to compute this by definition of conditional expectation. That is,

$$f_{X_{(1)}|X_{(2)}=x}(y_1) = \frac{f_{X_{(1)},X_{(2)}}(y_1, x)}{f_{X_{(2)}}(x)} = \frac{\frac{2}{a^2}e^{-y_1/a}\cdot e^{-x/a}}{\frac{2}{a}(1 - e^{-x/a})\cdot e^{-x/a}} = \frac{1}{a}\frac{e^{-y_1/a}}{1 - e^{-x/a}}$$

provided $0 < y_1 < x$. This then gives

$$E(X_{(1)}|X_{(2)} = x) = \int_{-\infty}^{\infty} f_{X_{(1)}|X_{(2)}=x}(y_1)\,dy_1 = \int_0^x \frac{y_1}{a}\frac{e^{-y_1/a}}{1 - e^{-x/a}}\,dy_1 = \frac{1}{a(1 - e^{-x/a})}\int_0^x y_1\,e^{-y_1/a}\,dy_1.$$

Integrating by parts gives

$$\int_0^x y_1\,e^{-y_1/a}\,dy_1 = a^2 - a^2e^{-x/a} - axe^{-x/a}.$$

Therefore,

$$E(X_{(1)}|X_{(2)}=x)=\frac{a^2-a^2e^{-x/a}-axe^{-x/a}}{a(1-e^{-x/a})}=a-\frac{xe^{-x/a}}{1-e^{-x/a}}=a-\frac{x}{e^{x/a}-1}.$$

Problem #10, page 116: Let X_1, X_2, and X_3 are independent, identically distributed $U(0,1)$ random variables. Notice that if $x>1/2$, then since $X_{(3)}>X_{(1)}$ we conclude

$$P(X_{(3)}>\tfrac{1}{2}\,|\,X_{(1)}=x)=1.$$

On the other hand, suppose that $0\le x\le 1/2$. By equation (3.10) on page 114,

$$f_{X_{(1)},X_{(3)}}(y_1,y_3)=6(y_3-y_1)$$

provided $0<y_1<y_3<1$. Therefore, we find

$$P(X_{(3)}>\tfrac{1}{2}\,|\,X_{(1)}=x)=\frac{\displaystyle\int_{1/2}^{1}f_{X_{(1)},X_{(3)}}(x,y_3)\,dy_3}{f_{X_{(1)}}(x)}.$$

For the numerator we calculate

$$\int_{1/2}^{1}f_{X_{(1)},X_{(3)}}(x,y_3)\,dy_3\int_{1/2}^{1}6(y_3-x)\,dy_3=(3y_3^2-6xy_3)\Big|_{1/2}^{1}=\frac{9}{4}-3x=\frac{3}{4}(3-4x).$$

As for the denominator, from Remark 3.1 on page 114, we find

$$f_{X_{(1)}}(x)=3(1-x)^2$$

provided $0<x<1$. Putting these pieces together, we conclude

$$P(X_{(3)}>\tfrac{1}{2}\,|\,X_{(1)}=x)=\frac{\frac{3}{4}(3-4x)}{3(1-x)^2}=\frac{(3-4x)}{4(1-x)^2}.$$

That is,

$$P(X_{(3)}>\tfrac{1}{2}\,|\,X_{(1)}=x)=\begin{cases}\frac{(3-4x)}{4(1-x)^2}, & \text{if } 0\le x\le 1/2,\\ 1, & \text{if } x>1/2.\end{cases}$$