Stat 351 Fall 2006
Assignment #5 Solutions

Problem #2, page 55: Suppose that $X + Y = c$. By definition of conditional density,

$$f_{X|X+Y=c}(x) = \frac{f_{X,X+Y}(x,c)}{f_{X+Y}(c)}.$$

We now find the joint density $f_{X,X+Y}(x,c)$. Let $U = X$ and $V = X + Y$ so that $X = U$ and $Y = V - U$. The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

Since X and Y are independent Exp(1), the joint density of (X,Y) is

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} e^{-x-y}, & \text{for } x > 0,\ y > 0, \\ 0, & \text{otherwise,} \end{cases}$$

The joint density of (U,V) is therefore given by

$$f_{U,V}(u,v) = f_{X,Y}(u, v-u) \cdot |J| = e^{-v}$$

provided that $u > 0$ and $v > u$. The marginal density for V is therefore

$$f_V(v) = \int_0^v e^{-v}\, du = ve^{-v}, \quad u > 0.$$

Since $V = X + Y$, we can rewrite these densities as $f_{X,X+Y}(x,c) = e^{-c}$, $0 < x < c$, and $f_{X+Y}(c) = ce^{-c}$, $c > 0$. Finally, we conclude

$$f_{X|X+Y=c}(x) = \frac{f_{X,X+Y}(x,c)}{f_{X+Y}(c)} = \frac{e^{-c}}{ce^{-c}} = \frac{1}{c}$$

provided that $0 < x < c$. That is,

$$X|X + Y = c \in U(0,c).$$

Problem #8, page 56: We begin with the observation that

$$Y_1|N = n \in \text{Bin}(n, 1/2).$$

Therefore, by the law of total probability, if $y = 0, 1, 2, \ldots$,

$$P(Y_1 = y) = \sum_{n=y}^{\infty} P(Y_1 = y|N = n)\, P(N = n) = \sum_{n=y}^{\infty} \binom{n}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{n-y} \cdot \frac{\lambda^n e^{-\lambda}}{n!}.$$

This can be manipulated in exactly the same way as in Example 3.2 on page 42. Thus, we conclude

$$P(Y_1 = y) = \frac{\lambda^y}{2^y y!} e^{-\lambda/2}, \quad \text{i.e.,} Y_1 \in \text{Po}(\lambda/2).$$

Next, since $Y_2 = N - Y_1$ we find that

$$P(Y_2 = y) = P(N - Y_1 = y) = P(Y_1 = N - y)$$

and so by the law of total probability, if $y = 0, 1, 2, \ldots$,

$$P(Y_2 = y) = \sum_{n=y}^{\infty} P(Y_1 = n - y | N = n) P(N = n)$$

But since $Y_1 | N = n \in \text{Bin}(n, 1/2)$, we find

$$P(Y_2 = y) = \sum_{n=y}^{\infty} \binom{n}{n-y} \left(\frac{1}{2}\right)^{n-y} \left(\frac{1}{2}\right)^{y} \cdot \frac{\lambda^n e^{-\lambda}}{n!} = \frac{\lambda^y}{2^y y!} e^{-\lambda/2}, \quad \text{i.e., } Y_2 \in \text{Po}(\lambda/2)$$

which also follows as in Example 3.2 on page 42. In order to show Y_1 and Y_2 are independent, we proceed as follows:

$$\begin{aligned} P(Y_2 = y_2 | Y_1 = y_1) = P(N - Y_1 = y_2 | Y_1 = y_1) &= P(N + y_1 + y_2 | Y_1 = y_1) \\ &= \frac{P(Y_1 = y_1 | N + y_1 + y_2)\, P(N = y_1 + y_2)}{P(Y_1 = y_1)} \end{aligned}$$

where the first equality follows since $Y_2 = N - Y_1$ and the last equality follows from Bayes' Theorem. We now know all of the required densities, and so substituting in gives

$$\begin{aligned} \frac{P(Y_1 = y_1 | N + y_1 + y_2)\, P(N = y_1 + y_2)}{P(Y_1 = y_1)} &= \binom{y_1 + y_2}{y_1} \left(\frac{1}{2}\right)^{y_1 + y_2} \frac{e^{-\lambda} \dfrac{\lambda^{y_1 + y_2}}{(y_1 + y_2)!}}{e^{-\lambda/2} \dfrac{(\lambda/2)^{y_1}}{y_1!}} \\ &= e^{-\lambda/2} \frac{(\lambda/2)^{y_2}}{y_2!} \\ &= P(Y_2 = y_2). \end{aligned}$$

That is, $P(Y_2 = y_2 | Y_1 = y_1) = P(Y_2 = y_2)$ and so Y_1 and Y_2 are independent.

Problem #9, page 56: **(a)** The density function for Y is given by

$$f_Y(y) = \int_0^{\infty} \frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}\, dx$$

provided that $0 < y < 1$. This can be integrated by parts twice to produce

$$f_Y(y) = \int_0^{\infty} \frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}\, dx = 1.$$

That is, $Y \in U(0, 1)$. However, a slicker proof uses the gamma function as follows. Let $u = -\frac{x}{y}$ so that $du = -\frac{1}{y}\, dx$, from which it follows that

$$f_Y(y) = \int_0^{\infty} \frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}\, dx = \frac{1}{2} \int_0^{\infty} u^2 e^{-u}\, du = \frac{1}{2} \Gamma(3) = \frac{2!}{2} = 1.$$

(b) The conditional density of X given $Y = y$ is therefore

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}}{1} = \frac{x^2}{2y^3} \cdot e^{-\frac{x}{y}}$$

provided that $x > 0$. That is, $X|Y = y \in \Gamma(3, y)$.

(c) Since $Y \in U(0,1)$, we know that $E(Y) = \frac{1}{2}$ and $\text{Var}(Y) = \frac{1}{12}$. We also use the fact from page 260 that the mean of a $\Gamma(p, a)$ random variable is pa and the variance is pa^2. Thus, we find that the mean of X is

$$E(X) = E(E(X|Y)) = E(3Y) = 3E(Y) = \frac{3}{2}$$

and the variance of X is

$$\begin{aligned}
\text{Var}(X) = \text{Var}(E(X|Y)) + E(\text{Var}(X|Y)) &= \text{Var}(3Y) + E(3Y^2) \\
&= 9\,\text{Var}(Y) + 3E(Y^2) \\
&= 9\,\text{Var}(Y) + 3\left[\text{Var}(Y) + (E(Y))^2\right] \\
&= \frac{9}{12} + 3\left(\frac{1}{12} + \frac{1}{4}\right) \\
&= \frac{7}{4}.
\end{aligned}$$

Problem #10, page 56: **(a)** Since

$$\int_0^1 \int_0^{1-x} cx\,dy\,dx = c \int_0^1 x(1-x)\,dx = c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{c}{6}$$

we conclude that $c = 6$.

(b) The marginal for Y is therefore given by

$$f_Y(y) = \int_0^{1-y} 6x\,dx = 3(1-y)^2, \quad 0 \le y \le 1$$

and the marginal for X is

$$f_X(x) = \int_0^{1-x} 6x\,dy = 6x(1-x), \quad 0 \le x \le 1.$$

We conditional densities are then

$$f_{X|Y=y}(x) = \frac{6x}{3(1-y)^2} = \frac{2x}{(1-y)^2}, \quad 0 \le x \le 1-y$$

and

$$f_{Y|X=x}(y) = \frac{6x}{6x(1-x)} = \frac{1}{1-x}, \quad 0 \le y \le 1-x.$$

Finally, we find

$$E(X|Y=y) = \int_0^{1-y} x \cdot \frac{2x}{(1-y)^2}\,dx = \frac{2}{3}(1-y)$$

and

$$E(Y|X=x) = \int_0^{1-x} y \cdot \frac{1}{1-x}\, dy = \frac{1}{2}(1-x).$$

Problem #19, page 58: If $X|M = m \in \text{Exp}(a)$ with $M^{-1} \in \Gamma(p,a)$, then in order to determine the distribution of X, the first step is to find the density of M. Thus, if $m > 0$,

$$F_M(m) = P(M \le m) = P\left(\frac{1}{M} \ge \frac{1}{m}\right) = \int_{1/m}^{\infty} f_{1/M}(x)\, dx.$$

Taking derivatives with respect to m gives

$$f_M(m) = \frac{1}{m^2} f_{1/M}(1/m) = \frac{1}{m^2} \cdot \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \cdot m^{1-p} \cdot e^{-1/(ma)} = \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \cdot m^{-1-p} \cdot e^{-1/(ma)}$$

provided that $m > 0$. Therefore, by the law of total probability,

$$\begin{aligned} f_X(x) = \int_0^\infty f_{X|M=m}(x) f_M(m)\, dm &= \int_0^\infty \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \cdot m^{-1-p} \cdot e^{-1/(ma)} \cdot \frac{1}{m} \cdot e^{-x/m}\, dm \\ &= \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \int_0^\infty m^{-2-p} \cdot e^{-\frac{1}{m}\left(x+\frac{1}{a}\right)}\, dm \end{aligned}$$

Let $u = \frac{1}{m}$ so that $du = -\frac{1}{m^2}\, dm$ and the integral above becomes

$$= \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \int_0^\infty u^p \cdot e^{-u\left(x+\frac{1}{a}\right)}\, du.$$

Next let $v = u\left(x + \frac{1}{a}\right)$ so that $dv = \left(x + \frac{1}{a}\right)\, du$ and the integral above becomes

$$= \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \cdot \left(x + \frac{1}{a}\right)^{-p-1} \int_0^\infty v^p \cdot e^{-v}\, dv.$$

But

$$\int_0^\infty v^p \cdot e^{-v}\, dv = \Gamma(p+1)$$

and so we conclude, for $x > 0$ (and using the fact that $\Gamma(p+1) = p \cdot \Gamma(p)$) that

$$f_X(x) = \frac{1}{\Gamma(p)} \cdot \frac{1}{a^p} \cdot \left(x + \frac{1}{a}\right)^{-p-1} \cdot \Gamma(p+1) = \frac{p}{a^p} \cdot \frac{1}{\left(x+\frac{1}{a}\right)^{p+1}}$$

which happens to be the density function of a translated Pareto distribution.