Solutions to Math 312 Midterm Exam – November 8, 2013

1. Since $i = e^{\pi i/2} = e^{\pi i/2 + 2\pi k i}$ for any $k \in \mathbb{Z}$, we see that if $e^{\zeta} = i$, then $\zeta \in \{\pi i/2 + 2\pi k i : k \in \mathbb{Z}\}$. Hence $z \in \{\pi/2 + 2\pi k : k \in \mathbb{Z}\}$.

2. Since $z'(t) = 3ie^{it}$, $0 \le t \le \pi/2$, by definition we have

$$\int_C \frac{1}{|z|^2}\,\mathrm{d}z = \int_0^{\pi/2} \frac{1}{|3e^{it}|^2} \cdot 3ie^{it}\,\mathrm{d}t = \frac{3i}{9}\int_0^{\pi/2} e^{it}\,\mathrm{d}t = \frac{1}{3}e^{it}\Big|_{t=0}^{t=\pi/2} = \frac{e^{i\pi/2}-1}{3} = \frac{i-1}{3}.$$

3. (Solution 1.) Observe that

$$\frac{z^2-4z+2}{z-2} = \frac{z^2-4z+4-2}{z-2} = \frac{(z-2)^2-2}{z-2} = (z-2) - \frac{2}{z-2}.$$

Since $f(z) = z-2$ is entire with continuous derivative $f'(z) = 1$, the Cauchy integral theorem implies $\int_C (z-2)\,\mathrm{d}z = 0$ and so

$$\int_C \frac{z^2-4z+2}{z-2}\,\mathrm{d}z = \int_C (z-2)\,\mathrm{d}z - 2\int_C \frac{1}{z-2}\,\mathrm{d}z = 0 - 2(2\pi i) = -4\pi i.$$

(Solution 2.) Observe that $f(z) = z^2 - 4z + 2$ is entire and that 2 is inside C. Thus, by the Cauchy integral formula,

$$\int_C \frac{z^2-4z+2}{z-2}\,\mathrm{d}z = \int_C \frac{f(z)}{z-2}\,\mathrm{d}z = 2\pi i f(2) = 2\pi i(2^2 - 4(2) + 2) = -4\pi i.$$

4. Observe that

$$\frac{z}{(z-1)(z-i)} = \frac{1/(1+i)}{z-1} + \frac{i/(1+i)}{z-i}$$

so that

$$\int_C \frac{z}{(z-1)(z-i)}\,\mathrm{d}z = \frac{1}{1+i}\int_C \frac{1}{z-1}\,\mathrm{d}z + \frac{i}{1+i}\int_C \frac{1}{z-i}\,\mathrm{d}z = \frac{2\pi i}{1+i} + \frac{2\pi i^2}{1+i} = \frac{2\pi(i-1)}{1+i} = 2\pi i.$$

5. Since $u(x,y) = x^2$ and $v(x,y) = y^2$, we find

$$u_x(x_0,y_0) = 2x_0, \quad u_y(x_0,y_0) = 0, \quad v_x(x_0,y_0) = 0, \quad v_y(x_0,y_0) = 2y_0.$$

Thus, the Cauchy-Riemann equations are satisfied at $z_0 = x_0 + y_0$ if and only if $x_0 = y_0$. Since u_x, u_y, v_x and v_y are all continuous along the line $y_0 = x_0$, we conclude that $f(z)$ is differentiable for any $z_0 \in \{z \in \mathbb{C} : z = x + ix\}$ with $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = 2x_0$.