Mathematics 312 (Fall 2013) September 20, 2013
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# Lecture #8: Powers and Roots of Algebraic Equations

**Example 8.1.** Find all values of $z \in \mathbb{C}$ such that $z^n = 1$ where $n$ is a positive integer.

**Solution.** Note that any solution will necessarily have modulus 1. Therefore, consider $\zeta = e^{i\varphi}$. There are $n$ values of $\varphi$ in $[0, 2\pi)$ for which

$$\zeta^n = e^{in\varphi} = 1$$

holds. In fact, they satisfy

$$n\varphi = 2k\pi, \quad k = 0, 1, \ldots, n-1$$

or, equivalently,

$$\varphi = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

Thus, the $n$ solutions are

$$\zeta_1 = e^{i0} = 1, \quad \zeta_2 = e^{i2\pi/n}, \quad \zeta_3 = e^{i4\pi/n}, \quad \ldots, \quad \zeta_n = e^{i2(n-1)\pi/n}.$$

We call

$$\{\zeta_1 = 1, \zeta_2, \ldots, \zeta_n\}$$

the $n$ *roots of unity*.

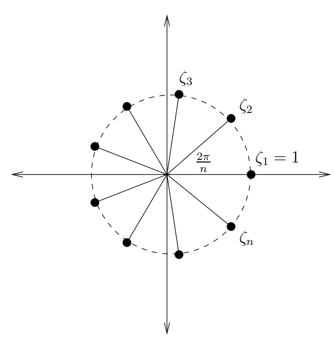


Figure 8.1: Geometric representation of the $n$ roots of unity.

We can represent the $n$ roots of unity as $n$ points equally spaces around the circle of radius 1. Note that $\zeta_1 = 1$, and each subsequent root is obtained by rotating the previous root by $2\pi/n$ radians. After $n$ rotations, we are back to our starting point.

**Exercise 8.2.** Determine all values of $z \in \mathbb{C}$ such that $z^5 = 1$.

**Answer.** $z_1 = 1$, $z_2 = e^{2\pi i/5}$, $z_3 = e^{4\pi i/5}$, $z_4 = e^{6\pi i/5}$, $z_5 = e^{8\pi i/5}$

**Example 8.3.** Determine all values of $z \in \mathbb{C}$ satisfying $z^4 = 16$.

**Solution.** We know that the fourth roots of unity are 1, $-1$, $i$, and $-i$. We know that the fourth root of the positive real number $|16|$ is 2. Thus, the possible fourth roots of the complex variable 16 are

$$\{2, -2, 2i, -2i\}.$$

**Example 8.4.** Suppose that $w \in \mathbb{C}$ is given. Determine all values of $z \in \mathbb{C}$ such that $z^n = w$.

**Solution.** Suppose that $\zeta$ is one such solution so that $\zeta^n = w$. If we write $w = re^{i\theta}$ and $\zeta = \rho e^{i\varphi}$, then we must have

$$\rho^n e^{in\varphi} = re^{i\theta}.$$

Since both $\rho$ and $r$ are non-negative real numbers, we must have $\rho = r^{1/n}$. Here we are writing $\rho$ for the unique positive real valued $n$th root of $r$. Moreover, since

$$\zeta = r^{1/n}e^{i\varphi} \quad \text{so that} \quad |\zeta| = r^{1/n}$$

we see that the solutions lie on the circle of radius $r^{1/n}$. Furthermore, we know that there are $n$ values of $\varphi \in [0, 2\pi)$ for which

$$e^{in\varphi} = e^{i\theta},$$

namely

$$n\varphi = \theta + 2k\pi, \quad k = 0, 1, \ldots, n-1,$$

or, equivalently,

$$\varphi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

Thus, the $n$ solutions to $z^n = w = re^{i\theta}$ are $\zeta_1, \ldots, \zeta_n$ where

$$\zeta_k = r^{1/n}e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, \ldots, n-1.$$

**Example 8.5.** Determine the two values of $z \in \mathbb{C}$ such that $z^2 = 2i$.

**Solution.** If we write $i$ in polar form as $e^{i\pi/2}$, then we conclude that the two complex square roots of $i$ are

$$e^{i\pi/4} \quad \text{and} \quad e^{5i\pi/4}.$$

Thus, the two required values of $z$ are

$$z_1 = \sqrt{2}e^{i\pi/4} = 1 + i \quad \text{and} \quad z_2 = \sqrt{2}e^{5i\pi/4} = -1 - i.$$