Lecture #7: Applications of Complex Exponentials

Definition. If $z = x + iy \in \mathbb{C}$, we define the *complex exponential* e^z as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Note that

$$|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x$$

since $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$ and $e^x > 0$ for $x \in \mathbb{R}$. In particular, if $\operatorname{Re}(z) \le 0$, then $|e^z| \le 1$.

Example 7.1. Express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin(3\theta)$.

Solution. We know from de Moivre's formula that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for any positive integer n and so

$$\sin(3\theta) = \operatorname{Im}[(\cos\theta + i\sin\theta)^3].$$

We know from the binomial theorem that

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

and so

$$(x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

Taking $x = \cos\theta$ and $y = \sin\theta$ yields

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

which in turn implies that

$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Substituting in $\sin^2\theta + \cos^2\theta = 1$ gives

$$\sin(3\theta) = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 3\sin^3\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

so that

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta).$$

nth Roots of a Complex Variable: An Application of Complex Exponentials

Example 7.2. Find all values of $z \in \mathbb{C}$ such that $z^4 = 1$.

Solution. Suppose that ζ is a solution to the equation. We begin by noting that $\zeta^4 = 1$ implies $|\zeta|^4 = 1$ which in turn implies $|\zeta| = 1$ so that ζ lies on the unit circle. Therefore, we assume that the polar form of ζ is $\zeta = e^{i\varphi}$ and so we need to solve

$$\zeta^4 = e^{i4\varphi} = e^{i0} = 1.$$

However, we know that

$$e^{i\varphi} = e^{i(\varphi+2k\pi)} \quad \text{for} \quad k \in \mathbb{Z}.$$

Since we want $\varphi \in [0, 2\pi)$, we conclude that

$$4\varphi \in \{0, 2\pi, 4\pi, 6\pi\} \quad \text{so that} \quad \varphi \in \{0, \pi/2, \pi, 3\pi/2\}.$$

Thus, there are four solutions to $z^4 = 1$, namely

$$\zeta_1 = e^{i0} = 1, \quad \zeta_2 = e^{i\pi/2} = i, \quad \zeta_3 = e^{i\pi} = -1, \quad \zeta_4 = e^{i3\pi/2} = -i.$$

We can plot these solutions in the complex plane.

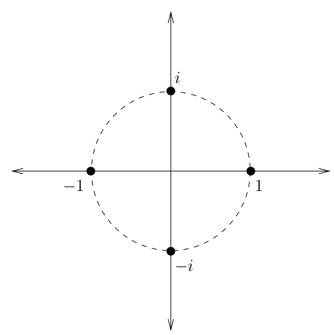

Figure 7.1: Geometric representation of solutions to $z^4 = 1$.

Also note that $\zeta_j^4 = 1$ for each $j = 1, 2, 3, 4$. Therefore, since multiplication of complex variables of unit modulus corresponds to rotation, we can conclude that the four roots are related to each other by a rotation of $2\pi/4 = \pi/2$ radians.

An equivalent way to think about the problem is as follows. We want to solve $z^4 = 1$ and so we know there will be four solutions. If we write 1 as $1 = e^{0i} = e^{2\pi i} = e^{4\pi i} = e^{6\pi i}$, then we see that the four solutions are

$$\zeta_1 = (e^{0i})^{1/4} = e^{i0} = 1, \quad \zeta_2 = (e^{2\pi i})^{1/4} = e^{i\pi/2} = i, \quad \zeta_3 = (e^{4\pi i})^{1/4} = e^{i\pi} = -1,$$

and

$$\zeta_4 = (e^{6\pi i})^{1/4} = e^{i3\pi/2} = -i.$$

Example 7.3. Find the three cube roots of 1; that is, determine all values of $z \in \mathbb{C}$ such that $z^3 = 1$.

Solution. Suppose that $\zeta = e^{i\varphi}$. We need to find the three values of $\varphi \in [0, 2\pi)$ such that

$$e^{i3\varphi} = 1.$$

If we write $1 = e^{i0}$, then the first value is $\varphi_1 = 0$ since

$$e^{i3\varphi_1} = e^{i0}.$$

If we write $1 = e^{i2\pi}$, then the second value is $\varphi_2 = 2\pi/3$ since

$$e^{i3\varphi_2} = e^{i2\pi}.$$

If we write $1 = e^{i4\pi}$, then the third value is $\varphi_3 = 4\pi/3$ since

$$e^{i3\varphi_3} = e^{i2\pi}.$$

Thus, the three solutions are

$$\zeta_1 = 1, \quad \zeta_2 = e^{i2\pi/3}, \quad \zeta_3 = e^{i4\pi/3}.$$

Note that we can write ζ_3 in polar form as $\zeta_3 = e^{-i2\pi/3}$. It is important to stress that in cartesian coordinates there is no ambiguity. The three cube roots of 1 are

$$\zeta_1 = 1, \quad \zeta_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \zeta_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$