Lecture #6: Applications of Complex Exponentials

Recall from last class that we defined the complex exponential $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using this we concluded that the polar form of $z \in \mathbb{C}$ can be written as

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$$

where $r = |z|$ and $\theta = \text{Arg}(z)$. We also proved that $z^n = r^n e^{in\theta}$ for any positive integer n. We will now use this to derive de Moivre's formula.

Theorem 6.1 (de Moivre's Formula)**.** *If n is a positive integer, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. Without loss of generality, assume that $\theta \in (-\pi, \pi]$ and consider $z = \cos\theta + i\sin\theta$ so that the polar form of z is $z = e^{i\theta}$. On the one hand we have

$$z^n = (\cos\theta + i\sin\theta)^n.$$

On the other hand we have

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

Equating the two gives

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

as required. □

We now observe that

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

If we solve this system of equations for $\cos\theta$ and $\sin\theta$, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Example 6.2. Find an identity for

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) \tag{$*$}$$

where n is a positive integer and $\theta \in \mathbb{R}$. Note that in the study of Fourier series it is important to be able to evaluate such an expression.

Before solving this problem, we need to establish a preliminary result. Recall the formula for a geometric series. If $x \in \mathbb{R}$ with $x \neq 1$, then

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

for any positive integer n. Moreover, if $|x| < 1$, then we can let $n \to \infty$ to obtain

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

Proposition 6.3. *If $z \in \mathbb{C}$ with $z \neq 1$, then*

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \tag{$**$}$$

for any positive integer n.

Proof. Since

$$(1 + z + z^2 + \cdots + z^n)(1 - z) = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}$$

and $z \neq 1$ we can divide by $(1 - z)$ to obtain the result. □

Solution. We can now find an identity for $(*)$. If we take $z = e^{i\theta}$ in $(**)$, we obtain

$$1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^n = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}}$$

or, equivalently,

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Taking the real parts of the previous express implies that

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) = \text{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right).$$

We will now find a simple expression for the right side of the previous equality. Note that

$$\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{1}{2i} \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2}}{\sin(\theta/2)}.$$

Now observe that

$$\begin{aligned} e^{i(n+\frac{1}{2})\theta} - e^{-i\theta/2} &= \left[\cos((n + \tfrac{1}{2})\theta) + i\sin((n + \tfrac{1}{2})\theta)\right] - \left[\cos(\theta/2) - i\sin(\theta/2)\right] \\ &= \cos((n + \tfrac{1}{2})\theta) - \cos(\theta/2) + i\left[\sin((n + \tfrac{1}{2})\theta) + \sin(\theta/2)\right] \end{aligned}$$

and so

$$
\begin{aligned}
\operatorname{Re}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right) &= \operatorname{Re}\left[\frac{1}{2i}\frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\theta/2}}{\sin(\theta/2)}\right] \\
&= \frac{1}{2\sin(\theta/2)}\operatorname{Re}\left[\tfrac{1}{i}\left(\cos((n+\tfrac{1}{2})\theta)-\cos(\theta/2)+i\left[\sin((n+\tfrac{1}{2})\theta)+\sin(\theta/2)\right]\right)\right] \\
&= \frac{1}{2\sin(\theta/2)}\left[\sin((n+\tfrac{1}{2})\theta)+\sin(\theta/2)\right]
\end{aligned}
$$

using the fact that $1/i = -i$. That is,

$$
1+\cos\theta+\cos(2\theta)+\cdots+\cos(n\theta) = \frac{\sin((n+\frac{1}{2})\theta)+\sin(\theta/2)}{2\sin(\theta/2)}.
$$