Lecture #3: Geometric Properties of $\mathbb{C}$

Recall that if $z = a + ib$ is a complex variable, then the modulus of $z$ is $|z| = \sqrt{a^2 + b^2}$ which may be interpreted geometrically as the distance from the origin to the point $(a, b) \in \mathbb{R}^2$. Since we can identify the complex variable $z \in \mathbb{C}$ with the point $(a, b) \in \mathbb{R}^2$, we conclude that $|z|$ represents the distance from $z$ to the origin.

**Example 3.1.** Describe the set $\{ z \in \mathbb{C} : |z| = 1 \}$.

**Solution.** Since $|z|$ represents the distance from the origin, the set $\{ z \in \mathbb{C} : |z| = 1 \}$ represents the set of all points that are at a distance 1 from the origin. This describes all points on the unit circle in the plane; see Figure 3.1.

![Figure 3.1: The set $\{ z \in \mathbb{C} : |z| = 1 \}$](image-url)

It is possible to derive this result analytically. If we let $z = x + iy$, then $|z|^2 = x^2 + y^2$. Since $|z| = 1$ if and only if $|z|^2 = 1$ if and only if $x^2 + y^2 = 1$, we conclude that

$$\{ z \in \mathbb{C} : |z| = 1 \} = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \},$$

the unit circle.

In general, we see that the set $\{ z \in \mathbb{C} : |z| = r \}$ describes a circle of radius $r$ centred at the origin. We can verify this using cartesian coordinates as follows. Suppose that $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Hence, $|z| = r$ if and only if $|z|^2 = r^2$, or equivalently, if and only if

$$x^2 + y^2 = r^2.$$

Moreover, if $z_0, z \in \mathbb{C}$, then one can easily verify that $|z - z_0|$ represents geometrically the distance from $z$ to $z_0$. This means that if $z_0 \in \mathbb{C}$ is given, then the set

$$\{ z \in \mathbb{C} : |z - z_0| = r \}$$

describes the circle of radius $r$ centred at $z_0$.

**Example 3.2.** Describe the set $\{ z \in \mathbb{C} : |z - i| = 2 \}$. 
Solution. If we write $i = 0 + i1$, then we see that $i$ corresponds to the point $(0, 1)$ in the plane. Therefore, the set in question represents a circle of radius 2 centred at $(0, 1)$.

![Figure 3.2: The set \{z ∈ C : |z − i| = 2\}.](image)

**Example 3.3.** Describe the set of $z ∈ C$ satisfying $|z + 2| = |z − 1|$.

**Solution.** Geometrically, $|z + 2|$ represents the distance from $z$ to $-2$, and $|z − 1|$ represents the distance from $z$ to 1. This means that we must find all $z ∈ C$ that are equidistant from both $-2$ and 1. If we view $-2$ as the point $(-2, 0)$ and 1 as the point $(1, 0)$, then we can easily conclude that the point $(-1/2, 0)$ is halfway between them. Thus, the point $-1/2$ belongs to the set \{z ∈ C : |z + 2| = |z − 1|\}. However, other points belong to this set. In fact, by drawing an isosceles triangle with altitude along the Re(z) = $-1/2$ line, we conclude that *any* point on the line Re(z) = $-1/2$ satisfies the condition $|z + 2| = |z − 1|$. This is described in Figure 3.3.

![Figure 3.3: The set \{z ∈ C : |z + 2| = |z − 1|\}.](image)

We can derive this result analytically as follows. Let $z = x + iy$ so that the condition $|z + 2| = |z − 1|$ is equivalent to $|z + 2|^2 = |z − 1|^2$ which in turn is equivalent to

$$(x + 2)^2 + y^2 = (x − 1)^2 + y^2.$$ 

Now $(x + 2)^2 = (x − 1)^2$ if and only if $x^2 + 4x + 4 = x^2 − 2x + 1$ if and only if $6x = −3$ which is, of course, equivalent to $x = −1/2$.

**Example 3.4.** Describe the set of $z ∈ C$ satisfying $|z − 1| = \text{Re}(z) + 1$. 

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Solution. In this case, it is easier to solve the problem analytically. If we write \( z = x + iy \), then \( |z - 1| = \text{Re}(z) + 1 \) is equivalent to \( |z - 1|^2 = (\text{Re}(z) + 1)^2 \) since \( |z - 1| = \text{Re}(z) + 1 \) is an equality between non-negative real numbers. Now, \( |z - 1|^2 = (x - 1)^2 + y^2 \) and \( (\text{Re}(z) + 1)^2 = (x + 1)^2 \) so that the set described is

\[
(x - 1)^2 + y^2 = (x + 1)^2.
\]

Now,

\[
y^2 = (x + 1)^2 - (x - 1)^2 = [(x + 1) + (x - 1)][(x + 1) - (x - 1)] = 4x
\]

(since \( (x + 1)^2 - (x - 1)^2 \) is a difference of perfect squares, this is easy to simplify) which represents a parabola parallel to the real axis as shown in Figure 3.4.

![Figure 3.4: The parabola \( y^2 = 4x \).](image)

Remark. In high school we do things like solve the equation \( |x + 2| = |x - 1| \) for \( x \). The solutions are points (i.e., real numbers). When we consider the same equation but in complex variables, \( |z + 2| = |z - 1| \), the solution is a curve in the complex plane. We can also see that the real solution, \(-1/2\) is one of the complex variables solutions of \( |z + 2| = |z - 1| \). However, we could have deduced this from the complex variables result. Here is how.

1. Consider the real equation that we wish to solve, namely \( |x + 2| = |x - 1| \) for \( x \in \mathbb{R} \).

2. Complexify the equation; that is, replace real variables by complex variables to obtain \( |z + 2| = |z - 1| \) for \( z \in \mathbb{C} \).

3. Determine the solutions to the complex variable problem; in this case, the answer is \( \text{Re}(z) = -1/2 \).

4. Since the solution must hold for all \( z \) satisfying the condition, it must necessarily hold for all \( z = x + i0 \in \mathbb{C} \) satisfying the condition. Thus, we see that the only \( z = x + i0 \in \mathbb{C} \) satisfying \( \text{Re}(z) = -1/2 \) is \( z = -1/2 \), and we conclude that the only solution to \( |x + 2| = |x - 1| \) for \( x \in \mathbb{R} \) is \( x = -1/2 \).

We will see many instances of this strategy in this course; in order to solve a real problem it will sometimes be easier to complexify, solve the complex variables problem, and extract the real solutions from the complex solutions.

Example 3.5. Describe the set of \( z \in \mathbb{C} \) satisfying \( z^2 + (\overline{z})^2 = 2 \).
Solution. Suppose that \( z = x + iy \) so that \( z^2 = (x + iy)^2 = x^2 - y^2 + i2xy \) and \( (\bar{z})^2 = (x - iy)^2 = x^2 - y^2 - i2xy \). This implies

\[
z^2 + (\bar{z})^2 = (x^2 - y^2 + i2xy) + (x^2 - y^2 - i2xy) = 2x^2 - 2y^2
\]

and so the set of \( z \in \mathbb{C} \) satisfying \( z^2 + (\bar{z})^2 = 2 \) is equivalent to the set

\[
\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}
\]

which describes a hyperbola as shown in Figure 3.5.

![Figure 3.5: The hyperbola \( x^2 - y^2 = 1 \).](image-url)