Mathematics 312 (Fall 2013) September 9, 2013
Prof. Michael Kozdron

Lecture #3: Geometric Properties of $\mathbb{C}$

Recall that if $z = a + ib$ is a complex variable, then the modulus of z is $|z| = \sqrt{a^2 + b^2}$ which may be interpreted geometrically as the distance from the origin to the point $(a, b) \in \mathbb{R}^2$. Since we can identify the complex variable $z \in \mathbb{C}$ with the point $(a, b) \in \mathbb{R}^2$, we conclude that $|z|$ represents the distance from z to the origin.

Example 3.1. Describe the set $\{z \in \mathbb{C} : |z| = 1\}$.

Solution. Since $|z|$ represents the distance from the origin, the set $\{z \in \mathbb{C} : |z| = 1\}$ represents the set of all points that are at a distance 1 from the origin. This describes all points on the unit circle in the plane; see Figure 3.1.

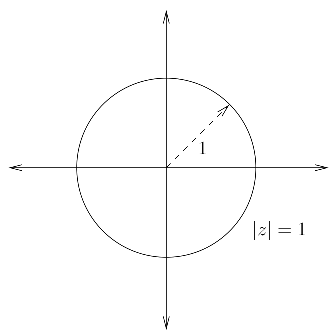

Figure 3.1: The set $\{z \in \mathbb{C} : |z| = 1\}$.

It is possible to derive this result analytically. If we let $z = x + iy$, then $|z|^2 = x^2 + y^2$. Since $|z| = 1$ if and only if $|z|^2 = 1$ if and only if $x^2 + y^2 = 1$, we conclude that

$$\{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

the unit circle.

In general, we see that the set $\{z \in \mathbb{C} : |z| = r\}$ describes a circle of radius r centred at the origin. We can verify this using cartesian coordinates as follows. Suppose that $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Hence, $|z| = r$ if and only if $|z|^2 = r^2$, or equivalently, if and only if

$$x^2 + y^2 = r^2.$$

Moreover, if $z_0, z \in \mathbb{C}$, then one can easily verify that $|z - z_0|$ represents geometrically the distance from z to z_0. This means that if $z_0 \in \mathbb{C}$ is given, then the set

$$\{z \in \mathbb{C} : |z - z_0| = r\}$$

describes the circle of radius r centred at z_0.

Example 3.2. Describe the set $\{z \in \mathbb{C} : |z - i| = 2\}$.

Solution. If we write $i = 0 + i1$, then we see that i corresponds to the point $(0, 1)$ in the plane. Therefore, the set in question represents a circle of radius 2 centred at $(0, 1)$.

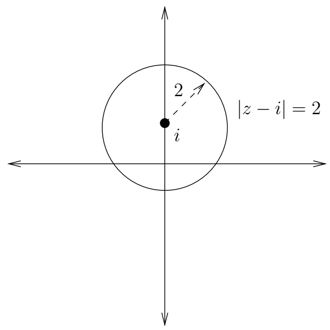

Figure 3.2: The set $\{z \in \mathbb{C} : |z - i| = 2\}$.

Example 3.3. Describe the set of $z \in \mathbb{C}$ satisfying $|z + 2| = |z - 1|$.

Solution. Geometrically, $|z+2|$ represents the distance from z to -2, and $|z-1|$ represents the distance from z to 1. This means that we must find all $z \in \mathbb{C}$ that are equidistant from both -2 and 1. If we view -2 as the point $(-2, 0)$ and 1 as the point $(1, 0)$, then we can easily conclude that the point $(-1/2, 0)$ is halfway between them. Thus, the point $-1/2$ belongs to the set $\{z \in \mathbb{C} : |z + 2| = |z - 1|\}$. However, other points belong to this set. In fact, by drawing an isosceles triangle with altitude along the $\text{Re}(z) = -1/2$ line, we conclude that *any* point on the line $\text{Re}(z) = -1/2$ satisfies the condition $|z + 2| = |z - 1|$. This is described in Figure 3.3.

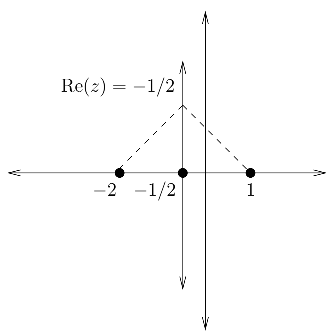

Figure 3.3: The set $\{z \in \mathbb{C} : |z + 2| = |z - 1|\}$.

We can derive this result analytically as follows. Let $z = x + iy$ so that the condition $|z + 2| = |z - 1|$ is equivalent to $|z + 2|^2 = |z - 1|^2$ which in turn is equivalent to

$$(x + 2)^2 + y^2 = (x - 1)^2 + y^2.$$

Now $(x+2)^2 = (x-1)^2$ if and only if $x^2 + 4x + 4 = x^2 - 2x + 1$ if and only if $6x = -3$ which is, of course, equivalent to $x = -1/2$.

Example 3.4. Describe the set of $z \in \mathbb{C}$ satisfying $|z - 1| = \text{Re}(z) + 1$.

Solution. In this case, it is easier to solve the problem analytically. If we write $z = x + iy$, then $|z - 1| = \text{Re}(z) + 1$ is equivalent to $|z - 1|^2 = (\text{Re}(z) + 1)^2$ since $|z - 1| = \text{Re}(z) + 1$ is an equality between non-negative real numbers. Now, $|z - 1|^2 = (x - 1)^2 + y^2$ and $(\text{Re}(z) + 1)^2 = (x + 1)^2$ so that the set described is

$$(x - 1)^2 + y^2 = (x + 1)^2.$$

Now,

$$y^2 = (x + 1)^2 - (x - 1)^2 = [(x + 1) + (x - 1)][(x + 1) - (x - 1)] = 4x$$

(since $(x + 1)^2 - (x - 1)^2$ is a difference of perfect squares, this is easy to simplify) which represents a parabola parallel to the real axis as shown in Figure 3.4.

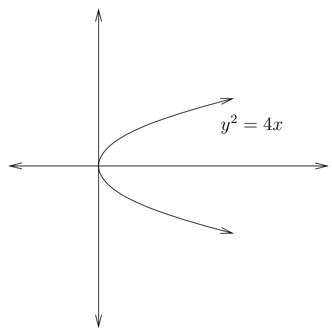

Figure 3.4: The parabola $y^2 = 4x$.

Remark. In high school we do things like solve the equation $|x + 2| = |x - 1|$ for x. The solutions are points (i.e., real numbers). When we consider the same equation but in complex variables, $|z + 2| = |z - 1|$, the solution is a curve in the complex plane. We can also see that the real solution, $-1/2$ is one of the complex variables solutions of $|z + 2| = |z - 1|$. However, we could have deduced this from the complex variables result. Here is how.

(1) Consider the real equation that we wish to solve, namely $|x + 2| = |x - 1|$ for $x \in \mathbb{R}$.

(2) Complexify the equation; that is, replace real variables by complex variables to obtain $|z + 2| = |z - 1|$ for $z \in \mathbb{C}$.

(3) Determine the solutions to the complex variable problem; in this case, the answer is $\text{Re}(z) = -1/2$.

(4) Since the solution must hold for all z satisfying the condition, it must necessarily hold for all $z = x + i0 \in \mathbb{C}$ satisfying the condition. Thus, we see that the only $z = x + i0 \in \mathbb{C}$ satisfying $\text{Re}(z) = -1/2$ is $z = -1/2$, and we conclude that the only solution to $|x + 2| = |x - 1|$ for $x \in \mathbb{R}$ is $x = -1/2$.

We will see many instances of this strategy in this course; in order to solve a real problem it will sometimes be easier to complexify, solve the complex variables problem, and extract the real solutions from the complex solutions.

Example 3.5. Describe the set of $z \in \mathbb{C}$ satisfying $z^2 + (\overline{z})^2 = 2$.

Solution. Suppose that $z = x + iy$ so that $z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$ and $(\overline{z})^2 = (x - iy)^2 = x^2 - y^2 - i2xy$. This implies

$$z^2 + (\overline{z})^2 = (x^2 - y^2 + i2xy) + (x^2 - y^2 - i2xy) = 2x^2 - 2y^2$$

and so the set of $z \in \mathbb{C}$ satisfying $z^2 + (\overline{z})^2 = 2$ is equivalent to the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$$

which describes a hyperbola as shown in Figure 3.5.

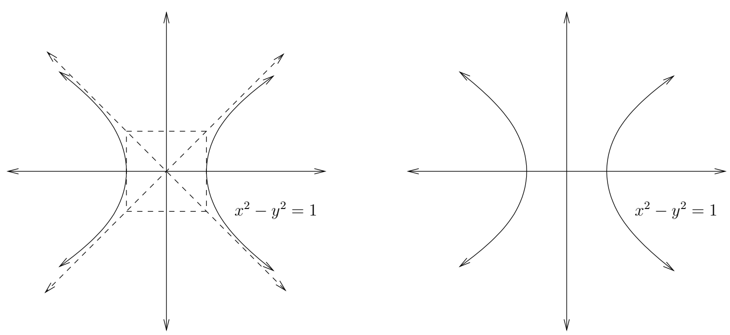

Figure 3.5: The hyperbola $x^2 - y^2 = 1$.