Lecture #34: Cauchy Principal Value

Definition. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on $(-\infty, \infty)$. If

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

exists, then we define the *Cauchy principal value of the integral of f over* $(-\infty, \infty)$ to be this value, and we write

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

for the value of this limit.

Remark. If

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

exists, then

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

However,

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

may exist, even though

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

does not exist. For instance,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\mathrm{d}x = 0 \quad \text{whereas} \quad \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\mathrm{d}x \text{ does not exist.}$$

We can now finish verifying Claim 2 from Example 33.1 of the previous lecture.

Example 33.1 (continued). Recall that we had deduced

$$\frac{\pi}{3} = \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z$$

where C_R is that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$. Taking the limit as $R \to \infty$ and using Theorem 33.2, we obtained

$$\frac{\pi}{3} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

By the definition of the Cauchy principal value, we have actually shown

$$\frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx.$$

We now observe that

$$\frac{x^2}{(x^2+1)(x^2+4)}$$

is an even function so that

$$\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = 2 \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx$$

which implies that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = 2 \lim_{R\to\infty} \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx.$$

In order to verify that the improper integral actually exists, note that

$$\left| \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx \right| = \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx \le \int_{0}^{R} \frac{1}{x^2+1} \, dx = \arctan R$$

using the inequality $x^2 \le (x^2+4)$. Since $\arctan R \to \pi/2$ as $R \to \infty$, we conclude

$$\lim_{R\to\infty} \int_{0}^{R} \frac{x^2}{(x^2+1)(x^2+4)} \, dx$$

exists by the integral comparison test. Thus,

$$\frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = 2 \int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx$$

so that

$$\int_{0}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} \, dx = \frac{\pi}{6}.$$

Example 34.1. Compute

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2+2x+1} \, dx.$$

Solution. Suppose that $C = C_R \oplus [-R, R]$ denotes the closed contour oriented counter-clockwise obtained by concatenating C_R, that part of the circle of radius R in the upper half plane parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$, with $[-R, R]$, the line segment along the real axis connecting the point $-R$ to the point R. Suppose further that

$$f(z) = \frac{1}{z^2+2z+2}$$

so that $f(z)$ has two simple poles. These occur where

$$z^2 + 2z + 2 = z^2 + 2z + 1 + 1 = (z+1)^2 + 1 = 0,$$

namely at $z_1 = i - 1$ and $z_2 = -i - 1 = -(i+1)$. Note that only z_1 is inside C, at least for R sufficiently large. Therefore, since

$$f(z) = \frac{1}{z^2 + 2z + 2} = \frac{1}{(z - z_1)(z - z_2)},$$

we conclude that

$$\operatorname{Res}(f; z_1) = \left. \frac{1}{z - z_2} \right|_{z=z_1} = \frac{1}{z_1 - z_2} = \frac{1}{i - 1 + (i+1)} = \frac{1}{2i}.$$

This implies

$$\int_C \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z = 2\pi i \frac{1}{2i} = \pi$$

so that

$$\begin{aligned} \pi = \int_C \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z &= \int_{[-R,R]} \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z + \int_{C_R} \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z \\ &= \int_{-R}^{R} \frac{1}{x^2 + 2x + 1} \, \mathrm{d}x + \int_{C_R} \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z. \end{aligned}$$

Taking $R \to \infty$ yields

$$\pi = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x^2 + 2x + 1} \, \mathrm{d}x + \lim_{R \to \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2} \, \mathrm{d}z = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 1} \, \mathrm{d}x$$

using Theorem 33.2 to conclude that the second limit is 0.