Lecture #34: Cauchy Principal Value

Definition. Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function on \((-\infty, \infty)\). If

\[
\lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx
\]

exists, then we define the \textit{Cauchy principal value of the integral of} \( f \) \textit{over} \((-\infty, \infty)\) to be this value, and we write

\[
\text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx
\]

for the value of this limit.

Remark. If

\[
\int_{-\infty}^{\infty} f(x) \, dx
\]

exists, then

\[
\int_{-\infty}^{\infty} f(x) \, dx = \text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx.
\]

However,

\[
\text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx
\]

may exist, even though

\[
\int_{-\infty}^{\infty} f(x) \, dx
\]

does not exist. For instance,

\[
\text{p.v.} \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, dx = 0 \quad \text{whereas} \quad \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, dx \quad \text{does not exist.}
\]

We can now finish verifying Claim 2 from Example 33.1 of the previous lecture.

Example 33.1 (continued). Recall that we had deduced

\[
\frac{\pi}{3} = \int_{-R}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx + \int_{C_R} \frac{z^2}{(z^2 + 1)(z^2 + 4)} \, dz
\]

where \( C_R \) is that part of the circle of radius \( R \) in the upper half plane parametrized by \( z(t) = Re^{it}, \ 0 \leq t \leq \pi \). Taking the limit as \( R \to \infty \) and using Theorem 33.2, we obtained

\[
\frac{\pi}{3} = \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx.
\]
By the definition of the Cauchy principal value, we have actually shown

\[ \frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx. \]

We now observe that

\[ \frac{x^2}{(x^2 + 1)(x^2 + 4)} \]

is an even function so that

\[ \int_{-R}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx = 2 \int_{0}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \]

which implies that

\[ \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx = 2 \lim_{R \to \infty} \int_{0}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx. \]

In order to verify that the improper integral actually exists, note that

\[ \left| \int_{0}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \right| = \int_{0}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \leq \int_{0}^{R} \frac{1}{x^2 + 1} \, dx = \arctan R \]

using the inequality \( x^2 \leq (x^2 + 4) \). Since \( \arctan R \to \pi/2 \) as \( R \to \infty \), we conclude

\[ \lim_{R \to \infty} \int_{0}^{R} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \]

exists by the integral comparison test. Thus,

\[ \frac{\pi}{3} = \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx = \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx = 2 \int_{0}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx \]

so that

\[ \int_{0}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{\pi}{6}. \]

**Example 34.1.** Compute

\[ \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 1} \, dx. \]

**Solution.** Suppose that \( C = C_R \oplus [-R, R] \) denotes the closed contour oriented counterclockwise obtained by concatenating \( C_R \), that part of the circle of radius \( R \) in the upper half plane parametrized by \( z(t) = Re^{it}, 0 \leq t \leq \pi \), with \([-R, R] \), the line segment along the real axis connecting the point \(-R\) to the point \( R \). Suppose further that

\[ f(z) = \frac{1}{z^2 + 2z + 2} \]
so that \( f(z) \) has two simple poles. These occur where

\[
z^2 + 2z + 2 = z^2 + 2z + 1 + 1 = (z + 1)^2 + 1 = 0,
\]

namely at \( z_1 = i - 1 \) and \( z_2 = -i - 1 = -(i + 1) \). Note that only \( z_1 \) is inside \( C \), at least for \( R \) sufficiently large. Therefore, since

\[
f(z) = \frac{1}{z^2 + 2z + 2} = \frac{1}{(z - z_1)(z - z_2)},
\]

we conclude that

\[
\text{Res}(f; z_1) = \frac{1}{z - z_2} \bigg|_{z=z_1} = \frac{1}{z_1 - z_2} = \frac{1}{i - 1 + (i + 1)} = \frac{1}{2i}.
\]

This implies

\[
\int_C \frac{1}{z^2 + 2z + 2} \, dz = 2\pi i \frac{1}{2i} = \pi
\]

so that

\[
\pi = \int_C \frac{1}{z^2 + 2z + 2} \, dz = \int_{[-R,R]} \frac{1}{z^2 + 2z + 2} \, dz + \int_{C_R} \frac{1}{z^2 + 2z + 2} \, dz
\]

\[
= \int_{-R}^{R} \frac{1}{x^2 + 2x + 1} \, dx + \int_{C_R} \frac{1}{z^2 + 2z + 2} \, dz.
\]

Taking \( R \to \infty \) yields

\[
\pi = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x^2 + 2x + 1} \, dx + \lim_{R \to \infty} \int_{C_R} \frac{1}{z^2 + 2z + 2} \, dz = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 1} \, dx
\]

using Theorem 33.2 to conclude that the second limit is 0.