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Lecture #30: Laurent Series and Residue Theory

A first look at residue theory as an application of Laurent series

One important application of the theory of Laurent series is in the computation of contour integrals. Suppose that $f(z)$ is analytic in the annulus $0 < |z - z_0| < R$ so that $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}.$$

Let C be any closed contour oriented counterclockwise lying entirely in the annulus and surrounding z_0 so that

$$\int_C f(z)\,\mathrm{d}z = \sum_{j=0}^{\infty} a_j \int_C (z - z_0)^j\,\mathrm{d}z + \sum_{j=1}^{\infty} a_{-j} \int_C (z - z_0)^{-j}\,\mathrm{d}z.$$

We know from Theorem 23.2 that

$$\int_C (z - z_0)^j\,\mathrm{d}z = \begin{cases} 2\pi i, & \text{if } j = -1, \\ 0, & \text{if } j \neq -1, \end{cases}$$

so that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i a_{-1}.$$

Thus, we see that the coefficient a_{-1} in the Laurent series expansion of $f(z)$ in an annulus of the form $0 < |z - z_0| < R$ is of particular importance. In fact, it has a name!

Definition. Suppose that the function $f(z)$ has an isolated singularity at z_0. The coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion of $f(z)$ around z_0 is called the *residue of $f(z)$ at z_0* and is denoted by

$$a_{-1} = \mathrm{Res}(f; z_0).$$

Classifying isolated singularities

We will now focus on functions that have an isolated singularity at z_0. Therefore, suppose that $f(z)$ is analytic in the annulus $0 < |z - z_0| < R$ and has an isolated singularity at z_0. Consider its Laurent series expansion

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}.$$

We call the part with the negative powers of $(z - z_0)$, namely

$$\sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j},$$

the *principal part* of the Laurent series. There are three mutually exclusive possibilities for the principal part.

- **(i)** If $a_j = 0$ for all $j < 0$, then we say that z_0 is a *removable singularity* of $f(z)$.
- **(ii)** If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$, but $a_j = 0$ for all $j < -m$, then we say that z_0 is a *pole of order* m for $f(z)$.
- **(iii)** If $a_j \neq 0$ for infinitely many $j < 0$, then we say that z_0 is an *essential singularity* of $f(z)$.

Example 30.1. Suppose that

$$f(z) = \frac{\sin z}{z}$$

for $|z| > 0$. Since the Laurent series expansion of $f(z)$ is

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

we conclude that $z_0 = 0$ is a removable singularity.

Example 30.2. Suppose that

$$f(z) = \frac{e^z}{z^m}$$

for $|z| > 0$ where m is a positive integer. Since

$$\begin{aligned} f(z) &= \frac{1}{z^m}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \\ &= \frac{1}{z^m} + \frac{1}{z^{m-1}} + \frac{1}{2!z^{m-2}} + \cdots + \frac{1}{(m-1)!z} + \frac{1}{m!} + \frac{z}{(m+1)!} + \cdots, \end{aligned}$$

we conclude that $z_0 = 0$ is a pole of order m.

Example 30.3. Suppose that

$$f(z) = e^{1/z}$$

for $|z| > 0$. Since

$$f(z) = e^{1/z} = 1 + (1/z) + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

we conclude that $z_0 = 0$ is an essential singularity.

Example 30.4. Suppose that $C = \{|z| = 1\}$ denotes the unit circle oriented counterclockwise. Compute the following three integrals:

(a) $\displaystyle\int_C \frac{\sin z}{z}\,\mathrm{d}z$,

(b) $\displaystyle\int_C \frac{e^z}{z^m}\,\mathrm{d}z$, where m is a positive integer, and

(c) $\displaystyle\int_C e^{1/z}\,\mathrm{d}z$.

Solution. In order to compute all three integrals, we use the fact that $z_0 = 0$ is an isolated singularity so that

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f;0).$$

(a) From Example 30.1, we know that $a_{-1} = 0$ so that

$$\int_C \frac{\sin z}{z}\,\mathrm{d}z = 0.$$

(b) From Example 30.2, we know that $a_{-1} = 1/(m-1)!$ so that

$$\int_C \frac{e^z}{z^m}\,\mathrm{d}z = \frac{2\pi i}{(m-1)!}.$$

(c) From Example 30.3, we know that $a_{-1} = 1$ so that

$$\int_C e^{1/z}\,\mathrm{d}z = 2\pi i.$$

Note that although we could have used the Cauchy Integral Formula to solve **(a)** and **(b)**, we could not have used it to solve **(c)**.

Question. Given the obvious importance of the coefficient a_{-1} in a Laurent series, it is natural to ask if there is any way to determine a_{-1} without computing the entire Laurent series.

Theorem 30.5. *A function $f(z)$ has a pole of order m at z_0 if and only if*

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$.

Proof. Suppose that $f(z)$ has a pole of order m at z_0. By definition, the Laurent series for $f(z)$ has the form

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \sum_{j=-(m-1)}^{\infty} a_j (z-z_0)^j$$

and so

$$(z-z_0)^m f(z) = a_{-m} + \sum_{j=-(m-1)}^{\infty} a_j (z-z_0)^{j+m} = a_{-m} + \sum_{j=1}^{\infty} a_{j-m}(z-z_0)^j.$$

Therefore, if we let

$$g(z) = a_{-m} + \sum_{j=1}^{\infty} a_{j-m}(z-z_0)^j,$$

then $g(z)$ is analytic in a neighbourhood of z_0. By assumption, $a_{-m} \neq 0$ since $f(z)$ has a pole of order m at z_0, and so $g(z_0) = a_{-m} \neq 0$.

Conversely, suppose that

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$. Since $g(z)$ is analytic, it can be expanded in a Taylor series about z_0, say

$$g(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots = \sum_{j=0}^{\infty} b_j (z-z_0)^j.$$

Since $g(z_0) = b_0 \neq 0$ by assumption, we obtain

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{j=0}^{\infty} b_j (z-z_0)^j = \frac{b_0}{(z-z_0)^m} + \frac{b_1}{(z-z_0)^{m-1}} + \cdots.$$

Therefore, by definition, $f(z)$ has a pole of order m at z_0. □

Now that we know the general form of a function $f(z)$ that has a pole of order m at z_0, we can determine a formula for $\operatorname{Res}(f; z_0)$, the residue of $f(z)$ at z_0, as follows.

Suppose that $f(z)$ has a pole of order m at z_0 so that from Theorem 30.5 we have

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

for some function $g(z)$ that is analytic in a neighbourhood of z_0 and has $g(z_0) \neq 0$. If C is a closed contour oriented counterclockwise containing z_0 and $f(z)$ is analytic inside and on C except for a pole of order m at z_0, then from our Laurent series development we have

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f; z_0). \tag{$\dagger$}$$

On the other hand, we can apply the Cauchy Integral Formula to conclude

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{g(z)}{(z-z_0)^m}\,\mathrm{d}z = 2\pi i \frac{g^{(m-1)}(z_0)}{(m-1)!}, \tag{‡}$$

so equating (†) and (‡) implies

$$\operatorname{Res}(f;z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Since $g(z) = (z-z_0)^m f(z)$, we conclude

$$\operatorname{Res}(f;z_0) = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_0)^m f(z)\bigg|_{z=z_0}.$$

Theorem 30.6. *If $f(z)$ is analytic for $0 < |z-z_0| < R$ and has a pole of order m at z_0, then*

$$\operatorname{Res}(f;z_0) = \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_0)^m f(z)\bigg|_{z=z_0} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-z_0)^m f(z).$$

In particular, if z_0 is a simple pole, then

$$\operatorname{Res}(f;z_0) = (z-z_0)f(z)\bigg|_{z=z_0} = \lim_{z\to z_0}(z-z_0)f(z).$$