Lecture #29: Calculating Laurent Series

Example 29.1. Determine the Laurent series for

$$f(z) = \frac{1}{(z-1)(2-z)}$$

for **(i)** $1 < |z| < 2$, and **(ii)** $|z| > 2$.

Solution. Note that the only singular points of $f(z)$ occur at 1 and 2. This means that (i) $f(z)$ is analytic in the annulus $1 < |z| < 2$, and (ii) $f(z)$ is analytic for $|z| > 2$. Hence, in either case, the Laurent series for $f(z)$ will necessarily be of the form

$$f(z) = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} z^{-j},$$

and so the idea is that we will find the coefficients a_j directly rather than by contour integration. (It is worth stressing that the coefficients in the Laurent series expansions for (i) and (ii) will not necessarily be the same.)

(i) Using partial fractions, we find

$$f(z) = \frac{1}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

Now consider

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{j=0}^{\infty} z^j.$$

We know this series converges for $|z| < 1$. However, we are interested in determining a series which converges for $|z| > 1$. Thus, we write

$$\frac{1}{z-1} = \frac{1/z}{1-1/z}$$

and observe that the series

$$\frac{1}{1-1/z} = \sum_{j=0}^{\infty} (1/z)^j$$

converges for $|1/z| < 1$, or equivalently, $|z| > 1$. This implies

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \frac{1}{z}\sum_{j=0}^{\infty} z^{-j} = \sum_{j=0}^{\infty} z^{-j-1} = \sum_{j=1}^{\infty} z^{-j} \quad \text{for } |z| > 1.$$

Now consider

$$-\frac{1}{z-2} = \frac{1/2}{1-z/2} = \frac{1}{2}\sum_{j=0}^{\infty} (z/2)^j = \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}}$$

which converges for $|z/2| < 1$, or equivalently, $|z| < 2$. Thus, when we add these two series, we obtain

$$\frac{1}{z-1} - \frac{1}{z-2} = \sum_{j=1}^{\infty} z^{-j} + \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}}.$$

Note that the first series converges for $|z| > 1$ while the second series converges for $|z| < 2$. This means that they BOTH converge when $1 < |z| < 2$, and so we have found the Laurent series for $f(z)$ for $1 < |z| < 2$, namely

$$f(z) = \frac{1}{(z-1)(2-z)} = \sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}} + \sum_{j=1}^{\infty} z^{-j}.$$

(ii) We now want the Laurent series for $f(z)$ to converge for $|z| > 2$. We already know that

$$\frac{1}{z-1} = \sum_{j=1}^{\infty} z^{-j}$$

converges for $|z| > 1$. However, the series that we found for

$$\frac{1}{z-2}$$

converges for $|z| < 2$. This means that we need to manipulate it differently, say

$$\frac{1}{z-2} = \frac{1/z}{1-2/z} = \frac{1}{z}\sum_{j=0}^{\infty}(2/z)^j = \sum_{j=0}^{\infty} 2^j z^{-j-1} = \sum_{j=1}^{\infty} 2^{j-1} z^{-j}$$

which converges for $|2/z| < 1$, or equivalently, $|z| > 2$. Thus,

$$f(z) = \frac{1}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{j=1}^{\infty} z^{-j} - \sum_{j=1}^{\infty} 2^{j-1} z^{-j} = \sum_{j=1}^{\infty} (1-2^{j-1}) z^{-j}$$

for $|z| > 2$.

Example 29.2. Determine the Laurent series for

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

for all $|z-1| > 0$.

Solution. Observe that $f(z)$ is analytic for $|z-1| > 0$. Now

$$f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(z-1+1)}}{(z-1)^3} = e^2 \frac{e^{2(z-1)}}{(z-1)^3}.$$

Recall that the Taylor series for e^w is

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{w^j}{j!}$$

which converges for all $w \in \mathbb{C}$. This implies

$$e^{2(z-1)} = 1 + 2(z-1) + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{2^j(z-1)^j}{j!}.$$

Hence,

$$\frac{e^{2(z-1)}}{(z-1)^3} = (z-1)^{-3} \sum_{j=0}^{\infty} \frac{2^j(z-1)^j}{j!} = \sum_{j=0}^{\infty} \frac{2^j(z-1)^{j-3}}{j!}$$

so that the Laurent series of $f(z)$ for $|z-1| > 0$ is

$$f(z) = \frac{e^{2z}}{(z-1)^3} = e^2 \sum_{j=0}^{\infty} \frac{2^j(z-1)^{j-3}}{j!} = e^2 \left(\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{2(z-1)} + \frac{8}{6} + \cdots \right).$$

Example 29.3. Let

$$\sinh z = \frac{e^z - e^{-z}}{2} = -i \sin(iz)$$

and let

$$f(z) = \frac{1}{z^2 \sinh z}.$$

Determine the first few terms of the Laurent series for $f(z)$ in $0 < |z| < \pi$, and then calculate

$$\int_C \frac{1}{z^2 \sinh z} \, \mathrm{d}z$$

where $C = \{|z| = 1\}$ is the unit circle centred at 0 oriented counterclockwise.

Solution. On Assignment #8 you showed that the Taylor series for the entire function $\sinh z$ is

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$$

which converges for all $z \in \mathbb{C}$. Moreover, it is not too difficult to show that $\sinh z = 0$ if and only if $z \in \{0, \pm i\pi, \pm 2i\pi, \ldots\}$. This implies that $f(z)$ is analytic for $0 < |z| < \pi$. Now

$$f(z) = \frac{1}{z^2 \sinh z} = \frac{1}{z^2 \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)} = \frac{1}{z^3} \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}.$$

Using the identity

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots,$$

we obtain

$$\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots} = 1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)^2 + \cdots = 1 - \frac{z^2}{6} + \frac{7z^4}{360} + \cdots$$

so that

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3}\left(1 - \frac{z^2}{6} + \frac{7z^4}{360} + \cdots\right) = \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots$$

for $0 < |z| < \pi$. Hence,

$$\int_C \frac{1}{z^2 \sinh z} \, dz = \int_C \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots \, dz = -\frac{2\pi i}{6} = -\frac{\pi i}{3}.$$

Example 29.4. Determine the Laurent series of

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

for $|z - 1| > 1$.

Solution. Note that $f(z)$ is analytic for $|z-1| > 1$ since the only singularity for $f(z)$ occurs at $z = 2$. Since we are expanding about the point $z_0 = 1$, the Laurent series will necessarily have the form

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j = \sum_{j=-\infty}^{\infty} a_j (z-1)^j.$$

Therefore, if we want to expand in powers of $z - 1$, we need to turn both our numerator and denominator into functions of $z - 1$. Observe that

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1} \frac{1}{1 - \frac{1}{z-1}} = \frac{1}{z-1} \sum_{j=0}^{\infty} \left(\frac{1}{z-1} \right)^j = \sum_{j=0}^{\infty} (z-1)^{-j-1} = \sum_{j=1}^{\infty} (z-1)^{-j}$$

for $|1/(z-1)| < 1$, or equivalently, $|z-1| > 1$, and

$$z^2 - 2z + 3 = (z-1)^2 + 2.$$

This yields

$$\begin{aligned} f(z) = \left[(z-1)^2 + 2 \right] \sum_{j=1}^{\infty} (z-1)^{-j} &= \sum_{j=1}^{\infty} \left(\frac{1}{z-1} \right)^{j-2} + 2 \sum_{j=1}^{\infty} \left(\frac{1}{z-1} \right)^j \\ &= (z-1) + 1 + \sum_{j=3}^{\infty} \left(\frac{1}{z-1} \right)^{j-2} + 2 \sum_{j=1}^{\infty} \left(\frac{1}{z-1} \right)^j \\ &= (z-1) + 1 + 3 \sum_{j=1}^{\infty} \left(\frac{1}{z-1} \right)^j \end{aligned}$$

for $|z - 1| > 1$ as the required Laurent series.