Mathematics 312 (Fall 2013) November 13, 2013
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Lecture #26: Taylor Series

Our primary goal for today is to prove that if $f(z)$ is an analytic function in a domain D, then $f(z)$ can be expanded in a Taylor series about any point $a \in D$. Moreover, the Taylor series for $f(z)$ converges uniformly to $f(z)$ for any z in a closed disk centred at a and contained entirely in D.

Theorem 26.1. *Suppose that $f(z)$ is analytic in the disk $\{|z-a| < R\}$. Then the sequence of Taylor polynomials for $f(z)$ about the point a, namely*

$$T_n(z; f, a) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(z-a)^j,$$

converges to $f(z)$ for all z in this disk. Furthermore, the convergence is uniform in any closed subdisk $\{|z-a| \le R' < R\}$. In particular, if $f(z)$ is analytic in $\{|z-a| < R\}$, then

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(z-a)^j. \qquad (\dagger)$$

We call $(\dagger)$ the *Taylor series for $f(z)$ about the point a.*

Proof. It is sufficient to prove uniform convergence in every subdisk $\{|z-a| \le R' < R\}$. Set $R'' = (R+R')/2$ and consider the closed contour $C = \{|z-a| = R''\}$ oriented counterclockwise. By the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta. \qquad (*)$$

Observe that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z-a)} = \frac{1}{\zeta - a} \frac{1}{1 - \left(\frac{z-a}{\zeta-a}\right)} = \frac{1}{\zeta - a} \frac{1}{1-w} \quad \text{where} \quad w = \left(\frac{z-a}{\zeta-a}\right)$$

and so using the fact that

$$\frac{1-w^{n+1}}{1-w} = 1 + w + w^2 + \cdots + w^n \quad \text{or equivalently} \quad \frac{1}{1-w} = 1 + w + \cdots + w^n + \frac{w^{n+1}}{1-w},$$

we conclude

$$\begin{aligned} \frac{1}{1 - \left(\frac{z-a}{\zeta-a}\right)} &= 1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\left(\frac{z-a}{\zeta-a}\right)^{n+1}}{1 - \left(\frac{z-a}{\zeta-a}\right)} \\ &= 1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta - a}{\zeta - z}\left(\frac{z-a}{\zeta-a}\right)^{n+1} \end{aligned}$$

and

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left[1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta-a}{\zeta-z}\left(\frac{z-a}{\zeta-a}\right)^{n+1}\right]. \qquad (**)$$

Substituting $(**)$ into $(*)$ we conclude

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-a}\left[1 + \left(\frac{z-a}{\zeta-a}\right) + \cdots + \left(\frac{z-a}{\zeta-a}\right)^n + \frac{\zeta-a}{\zeta-z}\left(\frac{z-a}{\zeta-a}\right)^{n+1}\right] \mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-a}\,\mathrm{d}\zeta + \frac{(z-a)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^2}\,\mathrm{d}\zeta + \cdots + \frac{(z-a)^n}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,\mathrm{d}\zeta \\ &\qquad + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta. \end{aligned}$$

However, from the Cauchy Integral Formula, we know

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-a}\,\mathrm{d}\zeta = f(a), \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^2}\,\mathrm{d}\zeta = f'(a), \quad \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^3}\,\mathrm{d}\zeta = \frac{f''(a)}{2!},$$

and in general

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^{j+1}}\,\mathrm{d}\zeta = \frac{f^{(j)}(a)}{j!}$$

so that

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta \\ &= T_n(z; f, a) + R_n(z; f, a). \end{aligned}$$

Thus, we see that in order to show that $T_n(z; f, a)$ converges to $f(z)$ uniformly for $|z-a| \le R'$, it suffices to show that $R_n(z; f, a)$ converges to 0 uniformly for $|z-a| \le R'$. Suppose, therefore, that $|z-a| \le R'$ and $|\zeta - a| = R''$ where $R'' = (R+R')/2$ as before. By the triangle inequality,

$$|\zeta - z| \ge R'' - R' = \frac{R+R'}{2} - R' = \frac{R-R'}{2},$$

and so

$$\begin{aligned} |R_n(z; f, a)| &= \left|\frac{(z-a)^{n+1}}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)(\zeta-a)^{n+1}}\,\mathrm{d}\zeta\right| \le \frac{1}{2\pi}\int_C \left|\frac{f(\zeta)(z-a)^{n+1}}{(\zeta-z)(\zeta-a)^{n+1}}\right| |\mathrm{d}\zeta| \\ &\le \frac{1}{2\pi}\int_C \frac{|f(\zeta)|(R')^{n+1}}{(R'')^{n+1}(R-R')/2}\,|\mathrm{d}\zeta| \\ &\le \frac{1}{\pi}\max_{\zeta\in C}|f(\zeta)|\left(\frac{R'}{R''}\right)^{n+1}\frac{1}{R-R'}\,\ell(C) \end{aligned}$$

where $\ell(C) = 2\pi R''$ is the arclength of C. That is, after some simplification, we obtain

$$|R_n(z; f, a)| \le \left(\frac{2R'}{R+R'}\right)^n \frac{2R'}{R-R'}\max_{\zeta\in C}|f(\zeta)|.$$

Notice that the right side of the previous inequality is independent of z. Since $2R' < R+R'$, the right side can be made less than any $\epsilon > 0$ by taking n sufficiently large. This gives the required uniform convergence. □

Example 26.2. Find the Taylor series for $f(z) = e^z$ about $a = 0$.

Solution. Since $f^{(n)}(z) = e^z$ so that $f^{(n)}(0) = 1$ for all non-negative integers n, we conclude

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

for every $z \in \mathbb{C}$.

Example 26.3. Find the Taylor series for both $f_1(z) = \sin z$ and $f_2(z) = \cos z$ about $a = 0$, and then show that the Taylor series for e^{iz} equals the sum of the Taylor series for $\cos z$ and $i \sin z$.

Solution. Observe that $f_1'(z) = \cos z = f_2(z)$ and $f_2'(z) = -\sin z = -f_1(z)$. Since $\cos 0 = 1$ and $\sin 0 = 0$, we obtain

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}$$

for every $z \in \mathbb{C}$. Observe that

$$\begin{aligned} e^{iz} &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \cdots \\ &= \left(1 + \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} + \frac{(iz)^6}{6!} + \cdots\right) + \left(iz + \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} + \frac{(iz)^7}{7!} + \cdots\right) \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z + \frac{i^2 z^3}{3!} + \frac{i^4 z^5}{5!} + \frac{i^6 z^7}{7!} + \cdots\right) \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= \cos z + i \sin z \end{aligned}$$

as expected. It is worth noting that term-by-term manipulations of the sum of Taylor series are justified by Theorem 26.1 since the Taylor series involved converge uniformly in closed disks about the point $a = 0$.

Remark. Sometimes the phrase *Maclaurin series* is used in place of Taylor series when $a = 0$.

Theorem 26.4. *If $f(z)$ is analytic at z_0, then the Taylor series for $f'(z)$ at z_0 can be obtained by termwise differentiation of the Taylor series for $f(z)$ about z_0 and converges in the same disk as the Taylor series for $f(z)$.*

Proof. Since $f(z)$ is analytic at z_0, the Taylor series for $f(z)$ about z_0 is given by

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

By termwise differentiation, we obtain

$$f'(z) = \sum_{j=0}^{\infty} j\frac{f^{(j)}(z_0)}{j!}(z - z_0)^{j-1} = \sum_{j=1}^{\infty} \frac{f^{(j)}(z_0)}{(j-1)!}(z - z_0)^{j-1}. \qquad (*)$$

Suppose now that $g(z) = f'(z)$. By Theorem 25.3, we know that $g(z)$ is analytic at z_0 so that its Taylor series is

$$g(z) = \sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!}(z - z_0)^j.$$

However, $g^{(j)}(z_0) = f^{(j+1)}(z_0)$ so that

$$f'(z) = g(z) = \sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z - z_0)^j. \qquad (**)$$

But by a change of index, it is clear that $(*)$ and $(**)$ are equal as required. □