Lecture #23: Applications of the Cauchy Integral Theorem

Last lecture we derived two results by direct calculation, namely

$$\int_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$$

where C is the circle of radius $r > 0$ centred at 0 oriented counterclockwise and, more generally,

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

for any $a \in \mathbb{C}$ with $|a| < r$. Note that the first result is a special case of the second result (i.e., with $a = 0$). Also note that the first result was relatively easy to derive whereas the second result was not.

Example 23.1. Suppose that $C_a = \{|z-a| < r\}$ denotes the circle of radius $r > 0$ centred at a oriented counterclockwise. Compute

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z.$$

Solution. Since the function

$$f(z) = \frac{1}{z-a}$$

is not analytic at a which happens to be inside C_a, we must evaluate this contour integral by definition. Let $z(t) = a + re^{it}$, $0 \le t \le 2\pi$, parametrize C so that $z'(t) = ire^{it}$. Therefore,

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{z(t)-a} z'(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{a+re^{it}-a}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

We have now determined by direct calculations that

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

where C is the circle of radius $r > 0$ centred at 0 oriented counterclockwise, C_a is the circle of radius $r > 0$ centred at a oriented counterclockwise, and $|a| < r$. We will now show that it is easy to determine

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

as a consequence of the fact that

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

which will render our horrendous calculation from Lecture #19 unnecessary. Consider Figure 23.1 below.

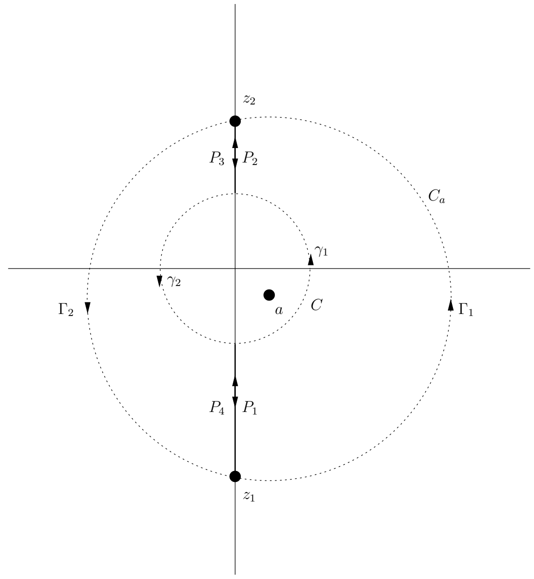

Figure 23.1: Continuous deformation of C into C_a.

Here we have taken z_1 and z_2 to be the points of intersection of C_a with the negative and positive imaginary axes, respectively. The curve Γ_1 connects z_1 to z_2 counterclockwise along C_a while the curve Γ_2 connects z_2 with z_1 counterclockwise along C_a. Note that

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = \int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Let P_1 be the curve that connects C_a to C along the negative imaginary axis, and let P_4 be the curve that connects C to C_a along the negative imaginary axis. Similarly, let P_2 be the curve that connects C to C_a along the positive imaginary axis, and let P_3 be the curve that connects C_a to C along the positive imaginary axis. Finally, let γ_1 be the curve counterclockwise along C connecting P_1 to P_2, and let γ_2 be the curve counterclockwise along C connecting P_3 to P_4. Note that

$$\int_{C} \frac{1}{z-a}\,\mathrm{d}z = \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Now here is the key. The function

$$f(z) = \frac{1}{z-a}$$

is analytic everywhere in $\mathbb{C}$ except at a. Therefore, the Fundamental Theorem of Calculus tells us that the value of the contour integral of $f(z)$ over any curve going from z_1 to z_2 is independent of the curve taken (as long as that curve does not pass through a). Now here are two curves going from z_1 to z_2, namely (i) Γ_1, and (ii) $P_1 \oplus \gamma_1 \oplus P_2$. This means

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_1\oplus\gamma_1\oplus P_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Similarly,

$$\int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_3\oplus\gamma_2\oplus P_4} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_3} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.$$

Adding these together gives

$$\begin{aligned}&\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z \\ &= \int_{P_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_3} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.\end{aligned}$$

However, since P_1 and P_4 follow the same path but in different directions, we have

$$\int_{P_1} \frac{1}{z-a}\,\mathrm{d}z = -\int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.$$

Similarly, P_2 and P_3 follow the same path but in the different directions so that

$$\int_{P_2} \frac{1}{z-a}\,\mathrm{d}z = -\int_{P_3} \frac{1}{z-a}\,\mathrm{d}z.$$

This implies

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

But we know

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z$$

and

$$\int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{C} \frac{1}{z-a}\,\mathrm{d}z$$

so that

$$\int_{C} \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i \qquad (*)$$

as desired.

Observe that by constructing an appropriate picture, we were able to *continuously deform* C to C_a and show that $(*)$ held. This leads to the following fact.

Theorem 23.2. *If C is a closed contour in the complex plane oriented counterclockwise and $a \in \mathbb{C}$ is in the interior of C, then*

$$\int_{C} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i.$$

Of course, Theorem 23.2 states that the same construction for $(*)$ holds for *any* contour C oriented counterclockwise surrounding the point a as shown in Figure 23.2 below.

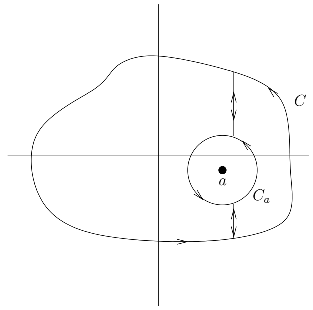

Figure 23.2: Continuous deformation of C into C_a.

Example 23.3. Compute
$$\int_C \frac{1}{z+i}\,\mathrm{d}z$$
where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Since $|-i| = 1 < 2$, we see that $a = -i$ is inside C so that
$$\int_C \frac{1}{z+i}\,\mathrm{d}z = 2\pi i.$$

Example 23.4. Compute
$$\int_C \frac{1}{2z+i}\,\mathrm{d}z$$
where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Since the integrand is not of the form $(z-a)^{-1}$, we cannot use the fact immediately. However,
$$\int_C \frac{1}{2z+i}\,\mathrm{d}z = \frac{1}{2}\int_C \frac{1}{z+i/2}\,\mathrm{d}z = \frac{1}{2}(2\pi i) = \pi i$$
since $a = -i/2$ is inside of the circle of radius 2 centred at 0.

Example 23.5. Compute
$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z$$
where C is the simple closed contour indicated in Figure 23.3 below.

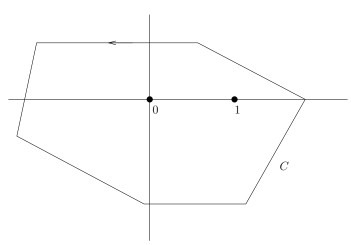

Figure 23.3: Figure for Example 23.5.

Solution. The trick is to use partial fractions on the integrand. That is,

$$\frac{3z-2}{z^2-z} = \frac{3z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

if and only if

$$A(z-1) + Bz = (A+B)z - A = 3z - 2.$$

This, of course, is true if and only if $A = 2$ and $B = 1$. That is,

$$\frac{3z-2}{z^2-z} = \frac{2}{z} + \frac{1}{z-1}$$

and so

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = \int_C \frac{2}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2(2\pi i) + 2\pi i = 6\pi i.$$

Example 23.6. Compute

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z$$

where C is the simple closed contour indicated in Figure 23.4 below.

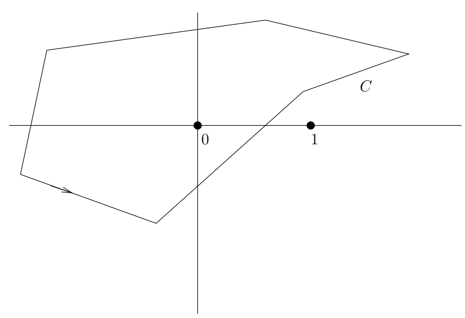

Figure 23.4: Figure for Example 23.6.

Solution. Again we can write

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z.$$

This time, however, $(z-1)^{-1}$ is analytic inside C since 1 is not inside C. The Cauchy Integral Theorem, Basic Version tells us that

$$\int_C \frac{1}{z-1}\,\mathrm{d}z = 0.$$

Therefore,

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2(2\pi i) + 0 = 4\pi i.$$

Example 23.7. Compute

$$\int_C \frac{1}{z^2-1}\,\mathrm{d}z$$

where C is the simple closed contour indicated in Figure 23.5 below.

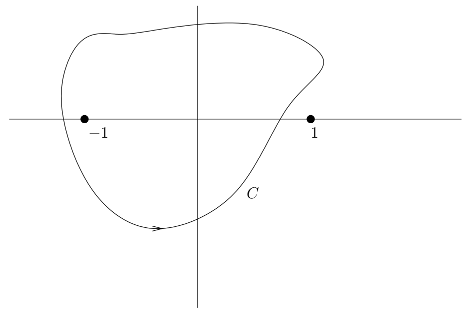

Figure 23.5: Figure for Example 23.7.

Solution. Using partial fractions, we find

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1/2}{z-1} - \frac{1/2}{z+1}.$$

Since $z = 1$ is not inside C, the Cauchy Integral Theorem, Basic Version tells us that

$$\int_C \frac{1}{z-1}\,\mathrm{d}z = 0.$$

Therefore,

$$\int_C \frac{1}{z^2-1}\,\mathrm{d}z = \frac{1}{2}\int_C \frac{1}{z-1}\,\mathrm{d}z - \frac{1}{2}\int_C \frac{1}{z+1}\,\mathrm{d}z = 0 - \frac{1}{2}(2\pi i) = -\pi i.$$