Lecture #20: Analyticity of the Complex Logarithm Function

Definition. Suppose that \( z \in \mathbb{C} \setminus \{0\} \). We define the principal value of the logarithm of \( z \), denoted \( \text{Log} z \), to be

\[
\text{Log} z = \log |z| + i \text{Arg}(z).
\]

Proposition 20.1. The function \( f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \) given by \( f(z) = \text{Log} z \) is continuous at all \( z \) except those along the negative real axis.

Proof. Since \( z \mapsto \log |z| \) is clearly continuous for all \( z \in \mathbb{C} \setminus \{0\} \) and since \( \text{Log} z = \log |z| + i \text{Arg}(z) \), the result follows from the fact that \( z \mapsto \text{Arg}(z) \) is discontinuous at each point on the nonpositive real axis. That is, let \( z = x_0 + iy \) for some \( x_0 < 0 \) fixed. If \( y \downarrow 0 \), then \( \text{Arg}(z) \downarrow \pi \), whereas if \( y \uparrow 0 \), then \( \text{Arg}(z) \uparrow -\pi \). \( \square \)

Recall that if \( f : (0, \infty) \rightarrow \mathbb{R} \) is given by \( f(x) = \log x \), then \( f'(x) = 1/x \). The same type of formula holds for the principal value of the logarithm, but must be stated very carefully.

Theorem 20.2. The function \( z \mapsto \text{Log} z \) is analytic in the domain \( D = \mathbb{C} \setminus D^* \) where

\[
D^* = \{ z \in \mathbb{C} : \text{Re}(z) \leq 0 \text{ and } \text{Im}(z) = 0 \}
\]

and satisfies

\[
\frac{d}{dz} \text{Log} z = \frac{1}{z}
\]

for \( z \in D \).

Proof. Let \( w = \text{Log} z \). We must show that

\[
\lim_{z \to z_0} \frac{w - w_0}{z - z_0}
\]

exists and equals \( 1/z_0 \) for every \( z_0 \in D \). However, we know (by definition of \( \text{Log} z \)) that \( z = e^w \). We also know from Example 15.2 that \( f(w) = e^w \) is entire with \( f'(w) = e^w \). In other words,

\[
\frac{d}{dw} f(w) \bigg|_{w=w_0} = \frac{d}{dw} e^w \bigg|_{w=w_0} = \frac{dz}{dw} \bigg|_{w=w_0} = \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = e^{w_0} = z_0.
\]

The next step is to observe that by continuity (Proposition 20.1), \( w \to w_0 \) as \( z \to z_0 \). Hence,

\[
\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{z - z_0}.
\]

However, compare the right side of \((**)\) with \((*)\) to conclude

\[
\frac{d}{dz} \text{Log} z \bigg|_{z=z_0} = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{1}{z_0} = \frac{1}{z_0}
\]

for every \( z_0 \in D \). \( \square \)
Remark. Assuming appropriate smoothness, we have shown that the real part of every analytic function $f$ is harmonic. The converse, however, is not true. That is, not every smooth harmonic function $u : D \to \mathbb{R}$ is necessarily the real part of some analytic function. As an example, consider $u(z) = \log|z|$ for $z \in D = \{0 < |z| < 1\}$. It is not hard to show that $u$ is harmonic in $D$. However, it can also be shown that $u$ does not have a harmonic conjugate in $D$. Compare this to Problem #10 on Assignment #4. The function $u(z) = \log|z|$ for $z \in D = \{\text{Re} \ z > r\}$ is harmonic in $D$ and does have a harmonic conjugate in $D$.

The Cauchy Integral Theorem

Our next goal is to investigate the conditions under which

$$\int_C f(z) \, dz = 0$$

for a closed contour $C$.

**Theorem 20.3** (Fundamental Theorem of Calculus for Integrals over Closed Contours). Suppose that $D$ is a domain. If $f(z)$ is continuous in $D$ and has an antiderivative $F(z)$ throughout $D$ (i.e., $F(z)$ is analytic in $D$ with $F'(z) = f(z)$ for every $z \in D$), then

$$\int_C f(z) \, dz = 0$$

for any closed contour $C$ lying entirely in $D$.

*Proof.* This follows from the usual Fundamental Theorem of Calculus. Suppose that $C$ is parametrized by $z = z(t)$, $a \leq t \leq b$. The hypothesis that $f(z)$ is continuous in $D$ is necessary for the contour integral

$$\int_C f(z) \, dz$$

to equal the Riemann integral

$$\int_a^b f(z(t)) \cdot z'(t) \, dt.$$

The assumption that $f$ has an antiderivative $F$ means that

$$\frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t).$$

Therefore,

$$\int_C f(z) \, dz = \int_a^b f(z(t)) \cdot z'(t) \, dt = \int_a^b \frac{d}{dt} F(z(t)) \, dt = F(z(b)) - F(z(a))$$

by the usual Fundamental Theorem of Calculus. The assumption that $C$ is a closed contour means that $z(a) = z(b)$ which implies $F(z(b)) = F(z(a))$. Hence,

$$\int_C f(z) \, dz = 0$$

for any closed contour $C$ lying entirely in $D$. 

**Remark.** This theorem can apply if $D$ is an annulus and $C$ surrounds the hole.