Lecture #19: Contour Integration

Example 19.1. Compute

\[ I_1 = \int_{C_1} \overline{z} \, dz \]

if \( C_1 = \{ e^{it}, 0 \leq t \leq \pi \} \) is that part of the upper half of the unit circle going from 1 to \(-1\).

Solution. If \( z(t) = e^{it}, 0 \leq t \leq \pi \), then \( z'(t) = ie^{it} \), and so

\[ \int_{C_1} \overline{z} \, dz = \int_0^\pi \overline{z(t)} \cdot z'(t) \, dt = \int_0^\pi e^{-it} \cdot ie^{it} \, dt = i \int_0^\pi dt = i\pi. \]

Example 19.2. Compute

\[ I_2 = \int_{C_2} \overline{z} \, dz \]

if \( C_2 = \{ e^{-it}, 0 \leq t \leq \pi \} \) is that part of the lower half of the unit circle going from 1 to \(-1\).

Solution. If \( z(t) = e^{-it}, 0 \leq t \leq \pi \), then \( z'(t) = -ie^{it} \), and so

\[ \int_{C_2} \overline{z} \, dz = \int_0^\pi \overline{z(t)} \cdot z'(t) \, dt = \int_0^\pi e^{it} \cdot -ie^{-it} \, dt = -i \int_0^\pi dt = -i\pi. \]

Note that the answers to the previous two examples are different; that is, even though the contours \( C_1 \) and \( C_2 \) start and end at the same points, \( I_1 \neq I_2 \). What is the difference between this pair of examples and the pair of examples from last lecture?

Theorem 19.3 (Fundamental Theorem of Calculus for Contour Integrals). Suppose that \( D \) is a domain. If \( f(z) \) is continuous in \( D \) and has an antiderivative \( F(z) \) throughout \( D \) (i.e., \( F(z) \) is analytic in \( D \) with \( F'(z) = f(z) \) for every \( z \in D \)), then

\[ \int_C f(z) \, dz = F(z(b)) - F(z(a)) \]

for any contour \( C \) lying entirely in \( D \).

Proof. Suppose that \( C \) lies entirely in \( D \) and is parametrized by \( z = z(t), a \leq t \leq b \). From the definition of contour integral, we have

\[ \int_C f(z) \, dz = \int_a^b f(z(t)) \cdot z'(t) \, dt \]

and note that the assumption that \( f(z) \) is continuous means that \( f(z(t)) \cdot z'(t) \) is Riemann integrable on \([a, b]\). The assumption that \( f \) has an antiderivative \( F \) means that

\[ \frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t). \]
Therefore,
\[
\int_C f(z) \, dz = \int_a^b f(z(t)) \cdot z'(t) \, dt = \int_a^b \frac{d}{dt} F(z(t)) \, dt = F(z(b)) - F(z(a))
\]
by the usual Fundamental Theorem of Calculus.

**Example 19.4.** Compute
\[
\int_C z^2 \, dz
\]
where \(C\) is any contour connecting 1 and \(2 + i\).

**Solution.** Observe that \(f(z) = z^2\) is continuous in \(C\) and \(F(z) = \frac{z^3}{3}\) is entire with \(F'(z) = f(z)\). Therefore, if \(C\) is any contour with \(z(a) = 1\) and \(z(b) = 2 + i\), then the Fundamental Theorem of Calculus for Contour Integrals implies
\[
\int_C z^2 \, dz = \frac{z^3}{3} \bigg|_{z=2+i} - \frac{z^3}{3} \bigg|_{z=1} = \left(\frac{(2+i)^3}{3} - \frac{1}{3}\right) = \frac{1}{3} + \frac{11}{3} \cdot i.
\]

**Remark.** This explains why the answers to Examples 18.4 and 18.5 are the same. Note that the function from Examples 19.1 and 19.2, namely \(\bar{z}\), does not have an antiderivative. This is why the Fundamental Theorem of Calculus for Contour Integrals does not apply, and so we are not surprised that contour integrals of \(\bar{z}\) do depend on the contour taken.

**Example 19.5.** Compute
\[
\int_C e^{iz} \, dz
\]
where \(C\) is that part of the unit circle in the first quadrant going from 1 to \(i\).

**Solution.** Observe that \(f(z) = e^{iz}\) is continuous in \(C\) and \(F(z) = -ie^{iz}\) is entire with \(F'(z) = f(z)\). Therefore, since \(C\) is a contour with \(z(a) = 1\) and \(z(b) = i\), the Fundamental Theorem of Calculus for Contour Integrals implies
\[
\int_C e^{iz} \, dz = -ie^{iz} \bigg|_{z=i} + ie^{iz} \bigg|_{z=1} = -ie - i = ie - ie^{-1}.
\]

The Complex Logarithm

Recall that we introduced the complex-valued logarithm function in Lecture #15. We will now re-visit that function. For real variables, we can define the *(natural) logarithm* of \(x > 0\), written as \(\log x\), to be that unique number satisfying \(e^{\log x} = x\). Moreover, we also know that \(\log(e^x) = x\) so that the functions \(f(x) = e^x\) and \(g(x) = \log x\) are inverses.

**Example 19.6.** Solve \(e^x = \pi/4\) for \(x \in \mathbb{R}\).

**Solution.** We can use logarithms to solve this problem. That is, \(e^x = \pi/4\) implies \(x = \log(e^x) = \log(\pi/4)\).
Remark. To solve the previous problem we used a key fact about real-valued logarithms, namely
\[ e^{x_1} = e^{x_2} \text{ if and only if } x_1 = x_2, \]
or, equivalently,
\[ \log x_1 = \log x_2 \text{ if and only if } x_1 = x_2. \]
We have already discovered that the function \( e^z \) is \( 2\pi i \) periodic, namely \( e^z = e^{z+2\pi i} \), so that we cannot simply define the complex-valued logarithm to be the inverse of \( e^z \).

Example 19.7. Solve \( e^z = 1 + i \) for \( z \in \mathbb{C} \).

Solution. We write \( 1 + i \) in polar coordinates as \( 1 + i = \sqrt{2}e^{i\pi/4} \) so that we need to solve
\[ e^z = \sqrt{2}e^{i\pi/4} \]
for \( z \). Consider \( e^\zeta = e^{i\pi/4} \). One solution is \( \zeta = i\pi/4 \). But this is not the only solution. By periodicity, we know \( e^\zeta = e^{\zeta+2\pi ki} \) for any \( k \in \mathbb{Z} \). Hence,
\[ e^{\zeta+2\pi ki} = e^{i\pi/4} \]
implies \( \zeta \in \{(\pi/4 + 2\pi k)i, \ k \in \mathbb{Z}\} \) and so
\[ z \in \left\{ \frac{1}{2}\log 2 + (\pi/4 + 2\pi k)i, \ k \in \mathbb{Z}\right\}. \]
Let \( w \in \mathbb{C} \neq 0 \). We know that there are infinitely many values of \( z \in \mathbb{C} \) such that \( e^z = w \); see Figure 19.1.

![Figure 19.1: The image of \( \mathbb{C} \) under the mapping \( e^z \).](image)

However, there is a unique value of \( z \) in the fundamental region \( \{ -\pi < \text{Im} z \leq \pi \} \) with \( e^z = w \). This is what we will use to define the logarithm of \( w \); more precisely, this will be the principal value of the logarithm.

Definition. Suppose that \( w \in \mathbb{C} \setminus \{0\} \). We define the principal value of the logarithm of \( w \), denoted \( \text{Log} w \), to be
\[ \text{Log} w = \log |w| + i \text{Arg}(w). \]
Remark. We are writing Log with a capital L to stress that it is the principal value of the complex-valued logarithm. Note that \( \log x \) for \( x \in \mathbb{R} \) denotes the usual real-valued natural logarithm.

Remark. The principal value of the logarithm of \( w \neq 0 \) can also be defined as the unique value of \( z \) with \(-\pi < \text{Im} z \leq \pi\) such that \( e^z = w \).

**Example 19.8.** Compute \( \log(1 + i) \).

**Solution.** Since \( |1 + i| = \sqrt{2} \) and \( \text{Arg}(1 + i) = \pi/4 \), we find

\[
\log(1 + i) = \log \sqrt{2} + i\pi/4 = \frac{1}{2} \log 2 + i \frac{\pi}{4}.
\]

**Definition.** Let \( w \in \mathbb{C} \setminus \{0\} \). The *complex-valued logarithm* of \( w \) is the multiple-valued function given by

\[
\log w = \log |w| + i \arg(w).
\]

Note that this is an equality of sets; since \( \arg(w) = \{ \text{Arg}(w) + 2\pi k, k \in \mathbb{Z} \} \), we can also write

\[
\log w = \{ \log |w| + i \text{Arg}(w) + 2\pi ki, k \in \mathbb{Z} \}.
\]

Recall from Assignment #1 that \( \arg(w_1w_2) = \arg(w_1) + \arg(w_2) \) for all \( w_1, w_2 \in \mathbb{C} \), but that \( \arg(w_1w_2) \neq \arg(w_1) + \arg(w_2) \) for all \( w_1, w_2 \in \mathbb{C} \). This translates into similar statements for the complex-valued logarithm and the principal value of the logarithm.

**Exercise 19.9.** Show that \( \log(w_1w_2) = \log w_1 + \log w_2 \) for all \( w_1, w_2 \in \mathbb{C} \setminus \{0\} \). Find values \( w_1, w_2 \in \mathbb{C} \setminus \{0\} \) such that \( \text{Log}(w_1w_2) \neq \text{Log} w_1 + \text{Log} w_2 \).