Mathematics 312 (Fall 2013) October 18, 2013
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Lecture #18: Contour Integration

A *contour integral* is just a two-dimensional line integral (also known as a path integral).

A *curve* in the complex plane will be denoted by $z = z(t)$, $a \leq t \leq b$. In other words, the function $z : \mathbb{R} \to \mathbb{C}$ given by $t \mapsto z(t)$, $a \leq t \leq b$, describes a curve in $\mathbb{C}$.

A *smooth curve* $z = z(t)$ is a curve such that

(i) $z(t)$ has a continuous derivative,

(ii) $z'(t) \neq 0$ for all $t \in [a, b]$,

(iii) $z(t)$ is a one-to-one function.

Example 18.1. In Figure 18.1 below are examples of a smooth curve (left) and a non-smooth curve (right).

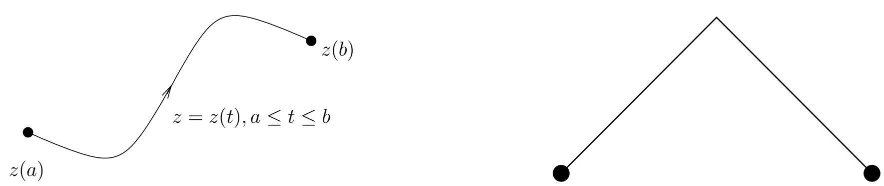

Figure 18.1: Figure for Example 18.1. The curve on the left is smooth, whereas the curve on the right is not smooth.

Example 18.2. Parametrize C_1, the line segment going from 1 to $2 + i$ in $\mathbb{C}$ as shown in Figure 18.2 below.

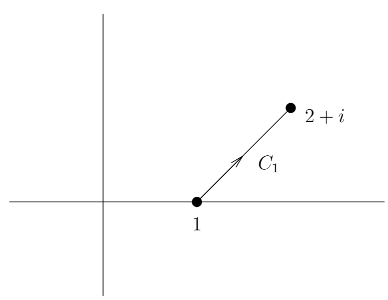

Figure 18.2: Figure for Example 18.2 and Example 18.4.

Solution. Let $x(t) = 1 + t$, $0 \leq t \leq 1$, and let $y(t) = t$, $0 \leq t \leq 1$, so that

$$z(t) = x(t) + iy(t) = 1 + t + it = 1 + (1 + i)t$$

for $0 \leq t \leq 1$.

Definition. A *contour* is a finite sequence of concatenated smooth curves $z = z(t)$ with a specified direction.

Example 18.3. Figure 18.3 below shows an example of a contour. Note that this particular contour is the concatenation of two smooth curves.

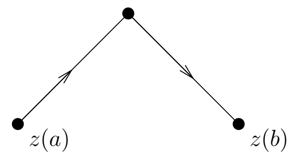

Figure 18.3: Figure for Example 18.3; an example of a contour.

Let C be a contour and consider

$$I = \int_C f(z)\,\mathrm{d}z$$

which is the contour integral of the function $f(z)$ along the contour C. That is, we integrate $f(z)$ along C in $\mathbb{C}$. Let $z = z(t)$, $a \le t \le b$, be a smooth parametrization of C. We define

$$I = \int_C f(z)\,\mathrm{d}z$$

to equal

$$I = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t.$$

Note that this definition requires

$$z'(t) = \frac{\mathrm{d}z(t)}{\mathrm{d}t}$$

to exist.

Example 18.4. Compute

$$I_1 = \int_{C_1} z^2\,\mathrm{d}z$$

where C_1 is the line segment going from 1 to $2+i$ in $\mathbb{C}$ as shown in Figure 18.2 above.

Solution. We know from Example 18.1 that C_1 is parametrized by $z(t) = 1 + (1+i)t$, $0 \le t \le 1$. Note that $z(0) = 1$ and $z(1) = 2+i$. Now

$$z(t)^2 = [1 + (1+i)t]^2 = 1 + 2(1+i)t + (1+i)^2t^2 \quad \text{and} \quad z'(t) = 1+i$$

so that

$$\begin{aligned}
\int_{C_1} z^2\,\mathrm{d}z = \int_0^1 z(t)^2 \cdot z'(t)\,\mathrm{d}t &= (1+i)\int_0^1 1 + 2(1+i)t + (1+i)^2t^2\,\mathrm{d}t \\
&= \left[(1+i)t + (1+i)^2t^2 + \frac{(1+i)^3}{3}t^3\right]_0^1 \\
&= (1+i) + (1+i)^2 + \frac{(1+i)^3}{3} \\
&= \frac{1}{3} + \frac{11}{3}i.
\end{aligned}$$

Example 18.5. Compute

$$I_{23} = \int_{C_{23}} z^2 \, \mathrm{d}z$$

where C_2 is the line segment going from 1 to 2 along the real axis, C_3 is the line segment going from 2 to $2+i$ parallel to the imaginary axis, and $C_{23} = C_2 \oplus C_3$ as shown in Figure 18.4 below.

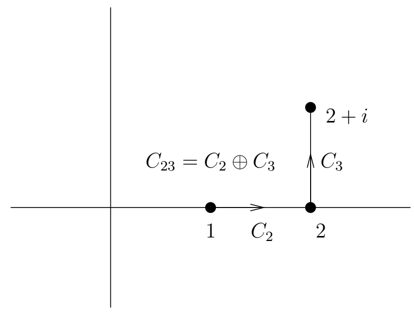

Figure 18.4: Figure for Example 18.5.

Solution. We can parametrize C_2 as follows. Let $x(t) = 1 + t$, $0 \le t \le 1$, and let $y(t) = 0$, $0 \le t \le 1$, so that

$$z(t) = x(t) + iy(t) = 1 + t$$

for $0 \le t \le 1$. We can parametrize C_3 as follows. Let $x(t) = 2$, $0 \le t \le 1$, and let $y(t) = t$, $0 \le t \le 1$, so that $z(t) = 2 + it$, $0 \le t \le 1$. Note, though, that we want to concatenate C_2 and C_3. Therefore, we will reparametrize C_3 by $z(t) = 2 + i(t-1) = 2 - i + it$, $1 \le t \le 2$, so that $C_{23} = C_2 \oplus C_3$ is parametrized by

$$z(t) = \begin{cases} 1 + t, & 0 \le t \le 1, \\ 2 - i + it, & 1 \le t \le 2. \end{cases}$$

Now,

$$\begin{aligned} \int_{C_{23}} z^2 \, \mathrm{d}z = \int_0^2 z(t)^2 \cdot z'(t) \, \mathrm{d}t &= \int_0^1 z(t)^2 \cdot z'(t) \, \mathrm{d}t + \int_1^2 z(t)^2 \cdot z'(t) \, \mathrm{d}t \\ &= \int_0^1 (1+t)^2 \cdot 1 \, \mathrm{d}t + \int_1^2 (2 - i + it)^2 \cdot i \, \mathrm{d}t \\ &= \frac{(1+t)^3}{3} \bigg|_0^1 + \frac{(2-i+it)^3}{3} \bigg|_1^2 \\ &= \frac{8}{3} - \frac{1}{3} + \frac{(2+i)^3}{3} - \frac{8}{3} \\ &= \frac{(2+i)^3}{3} - \frac{1}{3} \\ &= \frac{1}{3} + \frac{11}{3} i. \end{aligned}$$

Observe that our answers from Examples 18.4 and 18.5 are the same; that is,

$$I_1 = I_{23} = \frac{1}{3} + \frac{11}{3}i.$$

Is this a coincidence? In other words, we have taken two distinct contours connecting the same beginning and ending points, and found that the answer to both contour integrals is the same. Suppose we take more complicated contours connecting the same same beginning and ending points. Will we get the same value for any contour integral?