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Lecture #12: Limits, Continuity, and Differentiability

Definition. Let $f(z)$ be a function defined in some neighbourhood of z_0, except possibly at z_0 itself. We say that $f(z)$ *converges to* w_0 *as* z *converges to* z_0, written

$$\lim_{z \to z_0} f(z) = w_0,$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Definition. We say that $f(z)$ is *continuous at* z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Remark. This is the same definition as in calculus except that the condition $0 < |z - z_0| < \delta$ allows z to approach z_0 in any direction as shown in Figure 12.1. This makes limits much more subtle with complex variables.

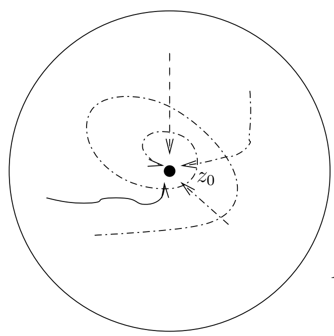

Figure 12.1: z can approach z_0 from any direction.

Definition. Let $f(z)$ be defined in a neighbourhood of z_0. The *derivative of* $f(z)$ *at* z_0 is

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z)\bigg|_{z_0} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided that the limit exists.

Remark. The limit must be independent of path $\Delta z \to 0$ in order for the derivative to exist.

Example 12.1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z$. Show that $f(z)$ is differentiable at z_0 for every $z_0 \in \mathbb{C}$.

Solution. Since

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z) - (z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

for all $z_0 \in \mathbb{C}$, we conclude that f is differentiable at z_0 for every $z_0 \in \mathbb{C}$ with $f'(z_0) = 1$.

Example 12.2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \overline{z}$. Is $f(z)$ differentiable at $z_0 \in \mathbb{C}$?

Solution. Observe that

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \overline{(z_0)}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z},$$

and so the question is to determine what happens as $\Delta z \to 0$. In particular, is the value independent of path? To see that it is not, let $\Delta z = \Delta x + i\Delta y$ so that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Consider approaching 0 along the positive real axis. This means that $\Delta y = 0$ so that $\Delta z = \Delta x$ and

$$\Delta z \to 0 \quad \text{if and only if} \quad \Delta x \to 0.$$

Therefore, we conclude

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0,\, \Delta y = 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Now consider approaching 0 along the positive imaginary axis. This means that $\Delta x = 0$ so that $\Delta z = i\Delta y$ and

$$\Delta z \to 0 \quad \text{if and only if} \quad \Delta y \to 0.$$

Therefore, we conclude

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x = 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Since the value of the limit is not independent of path, we conclude that $f(z) = \overline{z}$ is nowhere differentiable!

Example 12.3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = |z|^2$. Is $f(z)$ differentiable at $z_0 \in \mathbb{C}$?

Solution. Observe that

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} &= \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0\overline{z_0}}{\Delta z} \\ &= \frac{z_0\overline{z_0} + \overline{z_0}\Delta z + z_0\overline{\Delta z} + \overline{\Delta z}\Delta z - z_0\overline{z_0}}{\Delta z} \\ &= \overline{z_0} + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}\end{aligned}$$

and so the question is to determine what happens as $\Delta z \to 0$. Consider

$$\lim_{\Delta z \to 0} \left(\overline{z_0} + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right). \qquad (*)$$

We know that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \quad \text{does not exist}$$

and so the middle term in $(*)$ does not exist *except* when $z_0 = 0$. However, if $z_0 = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

This means that $f(z) = |z|^2$ is differentiable at $z_0 = 0$ with $f'(0) = 0$, but is not differentiable at any $z_0 \in \mathbb{C} \setminus \{0\}$.

Remark. The function $f(z) = \overline{z}$ is nowhere differentiable, and the function $f(z) = |z|^2$ is differentiable only at 0. As we will see more formally later, functions that involve $\overline{z}$ are typically not differentiable.

Definition. A function $f(z)$ is *analytic* in some domain D if it is differentiable at each point in D. (Recall that a domain is an open, connected set. In particular, D cannot be a single point.)

Definition. A function $f(z)$ is *analytic at* z_0 if it is differentiable at z_0 and if it is differentiable at all z in some neighbourhood of z_0.

Example 12.4. The function $f(z) = z$ is analytic at 0 since it is differentiable at 0 and is differentiable at all z in any neighbourhood of 0. (In fact, $f(z) = z$ is analytic in $\mathbb{C}$.) The function $f(z) = |z|^2$ is differentiable at 0, but it is not analytic at 0 since it is not differentiable at any $z \neq 0$.