Mathematics 312 (Fall 2013) September 27, 2013
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Lecture #11: Complex Functions as Mappings

Example 11.1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z^2$. Determine the image of the set $S = \{z \in \mathbb{C} : |z| \le 2,\ \text{Re}(z) = \text{Im}(z)\}$ under f.

Solution. We begin by observing that $\{\text{Re}(z) = \text{Im}(z)\}$ corresponds to the line $y = x$ in cartesian coordinates and $\{|z| \le 2\}$ represents the disk of radius 2 centred at the origin. Thus,

$$S = \{(x, y) \in \mathbb{R}^2 : y = x,\ x^2 + y^2 \le 2^2\}$$

or, equivalently in polar form,

$$S = \{z = re^{i\theta} : 0 \le r \le 2,\ \theta \in \{\pi/4, -3\pi/4\}\}.$$

Thus, if $z \in S$, then $z = re^{i\theta}$ so that $w = z^2 = r^2 e^{2i\theta} = \rho e^{i\varphi}$ where $\rho = r^2 \in [0, 4]$ and $\varphi = 2\theta \in \{\pi/2, -3\pi/2\}$. Of course, $\text{Arg}(e^{i\pi/2}) = \text{Arg}(e^{-3i\pi/2}) = \pi/2$ which implies that

$$f(S) = \{w = \rho e^{i\varphi} : 0 \le \rho \le 4,\ \varphi = \pi/2\}$$

which describes that part of the imaginary axis between 0 and 4; i.e.,

$$f(S) = \{w \in \mathbb{C} : \text{Re}(w) = 0,\ 0 \le \text{Im}(w) \le 4\}.$$

Example 11.2. Let $D = \{z \in \mathbb{C} : |z - 1| < 1\}$, and suppose that $f : D \to \mathbb{C}$ is given by $f(z) = 1/z$. Determine $f(D)$, the image of D under f.

Solution. Suppose that $w = f(z) = 1/z$. The condition $|z - 1| < 1$ is equivalent to the condition

$$\left|\frac{1}{w} - 1\right| < 1 \iff |1 - w| < |w|.$$

Hence, $f(D) = \{w \in \mathbb{C} : |1 - w| < |w|\}$. The problem with this description of $f(D)$ is that it is not at all clear which set of points is being described. Compare this to Problem #5 on Assignment #1. If we let $w = u + iv$, then

$$|1-w| < |w| \iff |1-w|^2 < |w|^2 \iff (u-1)^2+v^2 < u^2+v^2 \iff -2u+1 < 0 \iff u > \frac{1}{2}.$$

That is,

$$f(D) = \{w \in \mathbb{C} : \text{Re}(w) > 1/2\}.$$

Example 11.3. Let $D = \{z \in \mathbb{C} : -1 < \text{Re}(z) < 1\}$, and suppose that $f : D \to S$ is given by

$$f(z) = \frac{z + 1}{z - 1}.$$

Determine $f(D)$, the image of D under f.

Solution. Observe that we can write $w = f(z)$ as

$$w = f(z) = \frac{z+1}{z-1} = \frac{z-1+2}{z-1} = 1 + \frac{2}{z-1},$$

and so to determine $f(D)$ we simply break the calculation into separate steps since $f(z)$ is just the composition of 4 simple functions. It is

- a translation by -1, i.e., $z \mapsto z - 1$, followed by
- an inversion, i.e., $z \mapsto 1/z$, followed by
- an expansion, i.e., $z \mapsto 2z$, followed by
- a translation by $+1$, i.e., $z \mapsto z + 1$.

That is, if $h_1(z) = z - 1$, $h_2(z) = 1/z$, $h_3(z) = 2z$, and $h_4(z) = z + 1$, then

$$f(z) = h_4 \circ h_3 \circ h_2 \circ h_1(z) = h_4(h_3(h_2(h_1(z)))) = 1 + \frac{2}{z-1}.$$

Note that the image of h_1 is easy to determine. It is

$$S_2 = h_1(S) = \{z \in \mathbb{C} : -2 < \operatorname{Re}(z) < 0\}.$$

The image of S_2 under h_3 is a little more involved to compute. Let $w = 1/z$ and set $w = u+iv$ so that

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}..$$

The condition that $-2 < \operatorname{Re}(z) < 0$ is equivalent to $-2 < \operatorname{Re}(1/w) < 0$ and so u and v must satisfy

$$-2 < \frac{u}{u^2+v^2} < 0.$$

This means that we must have (i) $u < 0$ and (ii) $-2u^2 - 2v^2 < u$. Now observe that

$$-2u^2 - 2v^2 < u \iff 2u^2 + u + 2v^2 > 0 \iff 2u^2 + u + \frac{1}{8} + 2v^2 > \frac{1}{8}$$

$$\iff 2(u + \frac{1}{4})^2 + 2v^2 > \frac{1}{8} \iff \left(u + \frac{1}{4}\right)^2 + v^2 > \frac{1}{4^2}.$$

Thus, $S_3 = h_2(S_2)$ consists of those complex variables in quadrants II and III that are outside the disk of radius $1/4$ centred at $(-1/4, 0)$.

Next, we see that $S_4 = h_3(S_3)$ consists of those complex variables in quadrants II and III that are outside the disk of radius $1/2$ centred at $(-1/2, 0)$.

Finally, $f(S) = h_4(S_4)$ is obtained by translating S_4 by 1; that is, $f(S)$ consists of those points outside of the disk of radius $1/2$ centred at $(1/2, 0)$ having real part strictly less than 1. In other words,

$$f(S) = \{w \in \mathbb{C} : |w - 1/2| > 1/2,\ \operatorname{Re}(w) < 1\}.$$