University of Regina

Mathematics 312 – Complex Analysis I

Lecture Notes

Fall 2012

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Lecture #1: Introduction to Complex Variables

In calculus, we study

- algebraic operations with real numbers,
- functions, limits, continuity, graphing,
- differentiation and applications,
- integration and applications, and
- series and sequences.

In complex analysis, we will develop these topics in a parallel manner. Let $z = a + ib$, $i = \sqrt{-1}$, with a, $b \in \mathbb{R}$. We will study

- properties of the complex plane and algebraic operations with complex variables,
- properties of functions $f(z)$ with z complex, limits, graphing, differentiation, and
- integration of function $f(z)$ with z complex, say $\int_C f(z)\,\mathrm{d}z$ where C is some curve in the complex plane.

Algebra of Complex Variables

The motivation for introducing $i = \sqrt{-1}$ is to *solve* the equation $x^2 + 1 = 0$. In general, the fact that quadratic equations can have no real roots motivates introducing complex variables.

Notation. A *complex variable* z is of the form $z = a + ib$ where a and b are real numbers.

Definition. Two complex variables $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are *equal* if and only if $a_1 = a_2$ and $b_1 = b_2$.

Notation. Let $z = a + ib$ be a complex variable. The real part of z is $\mathrm{Re}(z) = a$ and the imaginary part of z is $\mathrm{Im}(z) = b$.

Fact. The complex variables z_1 and z_2 are equal iff $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

Note that we are using the phrase *complex variable* instead of *complex number*. This is because we wish to stress that $z = a+ib$ is *not* a number in the usual, or real, sense. Instead it is an object that we have created.

Arithmetic of Complex Variables

Let $i = \sqrt{-1}$, $z_1 = a_1 + ib_1$, and $z_2 = a_2 + ib_2$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}$. We define the operations of addition, multiplication, and division (provided either $a_2 \neq 0$ or $b_2 \neq 0$) as follows.

Addition. $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication. $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2)$

Division. $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2}\dfrac{a_2 - ib_2}{a_2 - ib_2} = \dfrac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\dfrac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}$

Remark. One way to remember these definitions is to manipulate the expressions just as you would for real numbers, but replacing i^2 by -1, i^3 by $-i$, and i^4 by 1. For example,

$$(a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 = a_1a_2 - b_1b_2 + i(b_1a_2 + a_1b_2).$$

The key is that the motivation for making the definitions we have comes from our experience with real numbers. However, there is no underlying reason why these expressions for addition, multiplication, and division of complex variables are true. They are simply definitions.

It can now be easily shown that if addition, multiplication, and division are defined in this way, then the following hold for complex variables z_1, z_2, z_3.

Commutative Law. $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$

Associative Law. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1z_2)z_3 = z_1(z_2z_3)$

Distributive Law. $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Exercise 1.1. Verify the commutative law, associative law, and distributive law hold for complex variables.

Remark. Consider the complex variable $z = a + ib$. We have $z = 0$ iff $a = 0$ and $b = 0$. Note that the complex variable 0 is shorthand for the complex variable $0 + i0$. Moreover, the real number a can be identified with the complex variable $a + i0$. We will, however, write this complex variable simply as a.

Proposition 1.2. *Consider the complex variables $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. If $z_1z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.*

Proof. Since $z_1z_2 = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2) = 0$ we conclude that

$$a_1a_2 - b_1b_2 = 0 \quad \text{and} \quad b_1a_2 + a_1b_2 = 0.$$

An equivalent way to write this system of equations is in matrix notation as

$$\begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In order to complete the proof, we will show that if $z_2 \neq 0$, then z_1 must be 0. Therefore, assume that $z_2 \neq 0$ so that either $a_2 \neq 0$ or $b_2 \neq 0$. In particular, this implies that

$$\det \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = a_2^2 + b_2^2 > 0.$$

However, recall from Math 122 that the only solution to the matrix system $A\mathbf{v} = \mathbf{0}$ with $\det A > 0$ is $\mathbf{v} = \mathbf{0}$. This implies that $a_1 = b_1 = 0$ so $z_1 = 0$ as required. $\square$

We end this discussion with one more convention concerning complex variables that is motivated by the arithmetic of real numbers. If k is a positive integer and z is a complex variable, then the power or exponential z^k is shorthand for multiplication of z by itself k times; for instance,

$$z^4 = zzzz.$$

We can then compute the product $zzzz$ using the associative law and the definition of multiplication of complex variables along with the identities $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. If m is a negative integer, say $m = -k$ for some non-negative integer k, define $z^m = z^{-k}$ by

$$z^{-k} = \frac{1}{z^k}.$$

Finally, take $z^0 = 1$.

Exercise 1.3. Verify that if $z = 1 + i$, then $z^4 = -4$.

Cartesian Representation (or Geometric Interpretation) of Complex Variables

We can represent the complex variable $z = a + ib$ as the point in the plane (a, b) as shown in Figure 1.1.

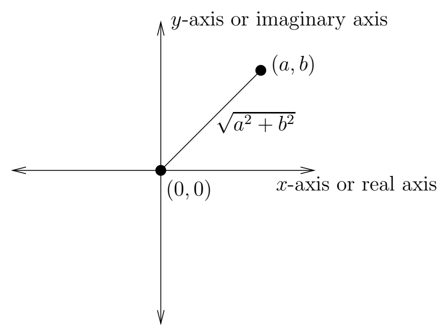

Figure 1.1: The identification of $\mathbb{C}$ with $\mathbb{R}^2$.

Note. In other words, if we let $\mathbb{C} = \{z = a+ib : a, b \in \mathbb{R}\}$ denote the set of complex variables, then we can identify $\mathbb{C}$ with the two-dimensional cartesian plane $\mathbb{R}^2$ via the identification

$$z = a + ib \in \mathbb{C} \longleftrightarrow (a, b) \in \mathbb{R}^2.$$

This identification is actually an *isomorphism* and so an algebraist might say that $\mathbb{C}$ and $\mathbb{R}^2$ are *isomorphic* and write $\mathbb{C} \cong \mathbb{R}^2$. We will not be concerned with isomorphisms in this class.

Observe that the distance from the point (a, b) in the plane to the origin $(0, 0)$ is

$$\sqrt{a^2 + b^2}.$$

This motivates the following definition.

Definition. Let $z = a + ib$ be a complex variable. The *modulus* or *absolute value* of z, denoted $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}.$$

Definition. Let $z = a + ib$ be a complex variable. The *(complex) conjugate* of z, denoted $\bar{z}$, is defined as

$$\bar{z} = a - ib.$$

Exercise 1.4. Suppose that z is a complex variable. Show that $z\bar{z} = |z|^2$.

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Lecture #2: Algebraic Properties of $\mathbb{C}$

Recall that $i = \sqrt{-1}$ and
$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$
denotes the set of complex variables. Also recall that if $z = a+ib \in \mathbb{C}$, then the modulus of z is $|z| = \sqrt{a^2 + b^2}$ and the conjugate of z is $\overline{z} = a - ib$. Geometrically, conjugation represents reflection in the real axis; see Figure 2.1.

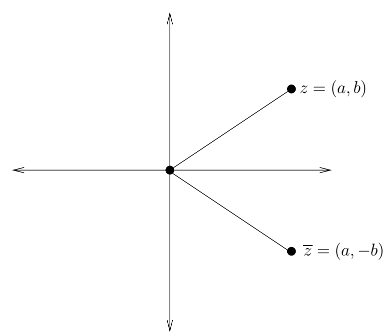

Figure 2.1: Geometric representation of complex conjugation.

Proposition 2.1. *If $z = a + ib$ is a complex variable, then $\sqrt{z\,\overline{z}}$ is a real number.*

Proof. Observe that
$$z\,\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2.$$
Since $|z|^2 = z\,\overline{z}$ is necessarily real and non-negative we can take square roots to obtain
$$\sqrt{z\,\overline{z}} = |z| = \sqrt{a^2 + b^2} \in \mathbb{R}$$
as required. $\square$

Proposition 2.2. *If z_1, $z_2 \in \mathbb{C}$, then $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.*

Proof. Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ so that
$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(b_1 a_2 + a_1 b_2)$$
implying that
$$\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(b_1 a_2 + a_1 b_2).$$
On the other hand,
$$\overline{z_1}\,\overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = a_1 a_2 - b_1 b_2 - ib_1 a_2 - ia_1 b_2 = (a_1 a_2 - b_1 b_2) - i(b_1 a_2 + a_1 b_2)$$
as well, and the proof is complete. $\square$

Exercise 2.3. Let z_1, $z_2 \in \mathbb{C}$. Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Exercise 2.4. Let $z \in \mathbb{C}$. Show that $\overline{(\overline{z})} = z$.

Before proving the next proposition, we observe the geometric interpretation of $|z|$, $\mathrm{Re}(z)$, and $\mathrm{Im}(z)$ as shown in Figure 2.2 below.

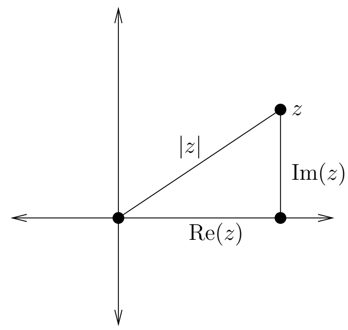

Figure 2.2: Geometric interpretation of $|z|$, $\mathrm{Re}(z)$, and $\mathrm{Im}(z)$.

Proposition 2.5. *If $z \in \mathbb{C}$, then*

(a) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$,

(b) $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$,

(c) $\mathrm{Re}(z) \leq |z|$, *and*

(d) $\mathrm{Im}(z) \leq |z|$.

Proof. Let $z = a + ib$ so that $\overline{z} = a - ib$. Solving the system of equations

$$z = a + ib \quad \text{and} \quad \overline{z} = a - ib$$

for a and b gives

$$a = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad b = \frac{1}{2i}(z - \overline{z}).$$

Moreover, since $|z| = \sqrt{a^2 + b^2}$, we see that

$$\mathrm{Re}(z) = a \leq \sqrt{a^2 + b^2} = |z| \quad \text{and} \quad \mathrm{Im}(z) = b \leq \sqrt{a^2 + b^2} = |z|$$

as required. □

Proposition 2.6. *If $z \in \mathbb{C}$, then $|\overline{z}| = |z|$.*

Proof. Let $z = a + ib$ so that $\overline{z} = a - ib$. Note that

$$|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

as required. □

Geometrically this proposition says that length doesn't change under a reflection through the real axis; see Figure 2.3.

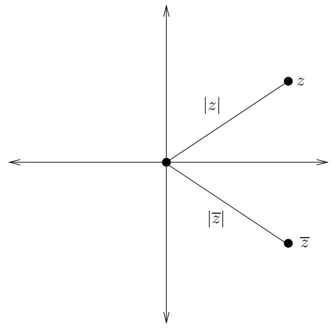

Figure 2.3: Geometric interpretation of $|\overline{z}| = |z|$.

Proposition 2.7. *If z_1, z_2 are complex variables, then $|z_1 z_2| = |z_1||z_2|$.*

Proof. Recall that $|w|^2 = w\,\overline{w}$ for any $w \in \mathbb{C}$. Let $w = z_1 z_2$ so that

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2(\overline{z_1}\,\overline{z_2}) = (z_1\,\overline{z_1})(z_2\,\overline{z_2}) = |z_1|^2|z_2|^2$$

using Proposition 2.2 and the Commutative Law. Since the moduli in question are non-negative real numbers we can take square roots to obtain

$$|z_1 z_2| = |z_1||z_2|$$

as required. $\square$

Proposition 2.8. *If z_1, z_2 are complex variables with $z_2 \neq 0$, then*

$$\frac{z_1}{z_2} = \frac{z_1\,\overline{z_2}}{|z_2|^2}.$$

Proof. Observe that

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\,\overline{z_2}}{|z_2|^2}$$

as required. $\square$

Theorem 2.9 (Triangle Inequality)**.** *If z_1, z_2 are complex variables, then*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. Recall that $|w|^2 = w\,\overline{w}$ for any $w \in \mathbb{C}$. Taking $w = z_1 + z_2$ implies

$$\begin{aligned} |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) &= z_1\,\overline{z_1} + z_2\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_2} \\ &= |z_1|^2 + |z_2|^2 + z_2\,\overline{z_1} + z_1\,\overline{z_2} \end{aligned}$$

using Exercise 2.3 and the Distributive Law. The next step is to deal with $z_2\,\overline{z_1} + z_1\,\overline{z_2}$. Recall that $2\,\mathrm{Re}(w) = w + \overline{w}$. If we take $w = z_1\,\overline{z_2}$, then

$$\overline{w} = \overline{z_1\,\overline{z_2}} = \overline{z_1}\,\overline{(\overline{z_2})} = \overline{z_1}\,z_2$$

using Proposition 2.2 and Exercise 2.4, and so we see that

$$z_2\,\overline{z_1} + z_1\,\overline{z_2} = \overline{w} + w = 2\,\mathrm{Re}(w) = 2\,\mathrm{Re}(z_1\,\overline{z_2}).$$

However, we also know from Proposition 2.5 that $\mathrm{Re}(w) \leq |w|$ which implies that

$$\mathrm{Re}(z_1\,\overline{z_2}) \leq |z_1\,\overline{z_2}| = |z_1||\overline{z_2}| = |z_1||z_2|.$$

Therefore, we conclude that

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2.$$

Since both sides of the inequality involve only non-negative real numbers, we can take square roots to obtain

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

as required. □

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Lecture #3: Geometric Properties of $\mathbb{C}$

Recall that if $z = a + ib$ is a complex variable, then the modulus of z is $|z| = \sqrt{a^2 + b^2}$ which may be interpreted geometrically as the distance from the origin to the point $(a, b) \in \mathbb{R}^2$. Since we can identify the complex variable $z \in \mathbb{C}$ with the point $(a, b) \in \mathbb{R}^2$, we conclude that $|z|$ represents the distance from z to the origin.

Example 3.1. Describe the set $\{z \in \mathbb{C} : |z| = 1\}$.

Solution. Since $|z|$ represents the distance from the origin, the set $\{z \in \mathbb{C} : |z| = 1\}$ represents the set of all points that are at a distance 1 from the origin. This describes all points on the unit circle in the plane; see Figure 3.1.

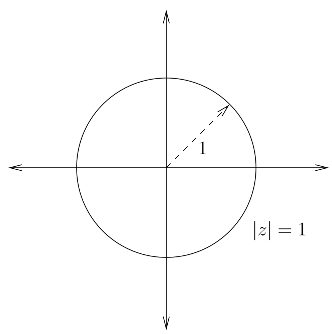

Figure 3.1: The set $\{z \in \mathbb{C} : |z| = 1\}$.

It is possible to derive this result analytically. If we let $z = x + iy$, then $|z|^2 = x^2 + y^2$. Since $|z| = 1$ if and only if $|z|^2 = 1$ if and only if $x^2 + y^2 = 1$, we conclude that

$$\{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

the unit circle.

In general, we see that the set $\{z \in \mathbb{C} : |z| = r\}$ describes a circle of radius r centred at the origin. We can verify this using cartesian coordinates as follows. Suppose that $z = x + iy$ so that $|z| = \sqrt{x^2 + y^2}$. Hence, $|z| = r$ if and only if $|z|^2 = r^2$, or equivalently, if and only if

$$x^2 + y^2 = r^2.$$

Moreover, if $z_0, z \in \mathbb{C}$, then one can easily verify that $|z - z_0|$ represents geometrically the distance from z to z_0. This means that if $z_0 \in \mathbb{C}$ is given, then the set

$$\{z \in \mathbb{C} : |z - z_0| = r\}$$

describes the circle of radius r centred at z_0.

Example 3.2. Describe the set $\{z \in \mathbb{C} : |z - i| = 2\}$.

Solution. If we write $i = 0 + i1$, then we see that i corresponds to the point $(0, 1)$ in the plane. Therefore, the set in question represents a circle of radius 2 centred at $(0, 1)$.

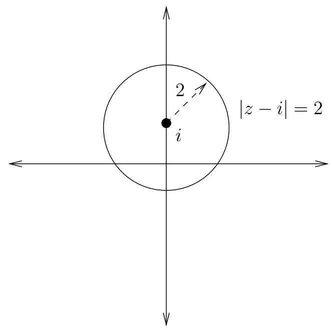

Figure 3.2: The set $\{z \in \mathbb{C} : |z - i| = 2\}$.

Example 3.3. Describe the set of $z \in \mathbb{C}$ satisfying $|z + 2| = |z - 1|$.

Solution. Geometrically, $|z+2|$ represents the distance from z to -2, and $|z-1|$ represents the distance from z to 1. This means that we must find all $z \in \mathbb{C}$ that are equidistant from both -2 and 1. If we view -2 as the point $(-2, 0)$ and 1 as the point $(1, 0)$, then we can easily conclude that the point $(-1/2, 0)$ is halfway between them. Thus, the point $-1/2$ belongs to the set $\{z \in \mathbb{C} : |z + 2| = |z - 1|\}$. However, other points belong to this set. In fact, by drawing an isosceles triangle with altitude along the $\mathrm{Re}(z) = -1/2$ line, we conclude that *any* point on the line $\mathrm{Re}(z) = -1/2$ satisfies the condition $|z + 2| = |z - 1|$. This is described in Figure 3.3.

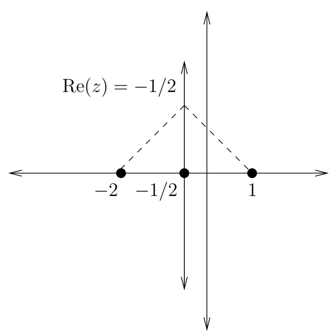

Figure 3.3: The set $\{z \in \mathbb{C} : |z + 2| = |z - 1|\}$.

We can derive this result analytically as follows. Let $z = x + iy$ so that the condition $|z + 2| = |z - 1|$ is equivalent to $|z + 2|^2 = |z - 1|^2$ which in turn is equivalent to

$$(x + 2)^2 + y^2 = (x - 1)^2 + y^2.$$

Now $(x+2)^2 = (x-1)^2$ if and only if $x^2+4x+4 = x^2-2x+1$ if and only if $6x = -3$ which is, of course, equivalent to $x = -1/2$.

Example 3.4. Describe the set of $z \in \mathbb{C}$ satisfying $|z - 1| = \mathrm{Re}(z) + 1$.

Solution. In this case, it is easier to solve the problem analytically. If we write $z = x + iy$, then $|z-1| = \text{Re}(z) + 1$ is equivalent to $|z-1|^2 = (\text{Re}(z)+1)^2$ since $|z-1| = \text{Re}(z)+1$ is an equality between non-negative real numbers. Now, $|z-1|^2 = (x-1)^2 + y^2$ and $(\text{Re}(z)+1)^2 = (x+1)^2$ so that the set described is

$$(x-1)^2 + y^2 = (x+1)^2.$$

Now,

$$y^2 = (x+1)^2 - (x-1)^2 = [(x+1)+(x-1)][(x+1)-(x-1)] = 2x$$

(since $(x+1)^2 - (x-1)^2$ is a difference of perfect squares, this is easy to simplify) which represents a parabola parallel to the real axis as shown in Figure 3.4.

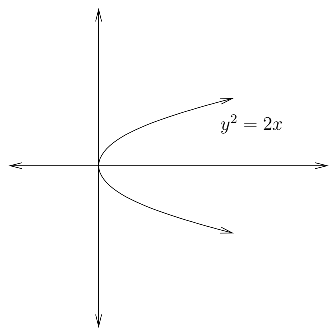

Figure 3.4: The parabola $y^2 = 2x$.

Remark. In high school we do things like solve the equation $|x+2| = |x-1|$ for x. The solutions are points (i.e., real numbers). When we consider the same equation but in complex variables, $|z+2| = |z-1|$, the solution is a curve in the complex plane. We can also see that the real solution, $-1/2$ is one of the complex variables solutions of $|z+2| = |z-1|$. However, we could have deduced this from the complex variables result. Here is how.

(1) Consider the real equation that we wish to solve, namely $|x+2| = |x-1|$ for $x \in \mathbb{R}$.

(2) Complexify the equation; that is, replace real variables by complex variables to obtain $|z+2| = |z-1|$ for $z \in \mathbb{C}$.

(3) Determine the solutions to the complex variable problem; in this case, the answer is $\text{Re}(z) = -1/2$.

(4) Since the solution must hold for all z satisfying the condition, it must necessarily hold for all $z = x + i0 \in \mathbb{C}$ satisfying the condition. Thus, we see that the only $z = x + i0 \in \mathbb{C}$ satisfying $\text{Re}(z) = -1/2$ is $z = -1/2$, and we conclude that the only solution to $|x+2| = |x-1|$ for $x \in \mathbb{R}$ is $x = -1/2$.

We will see many instances of this strategy in this course; in order to solve a real problem it will sometimes be easier to complexify, solve the complex variables problem, and extract the real solutions from the complex solutions.

Example 3.5. Describe the set of $z \in \mathbb{C}$ satisfying $z^2 + (\overline{z})^2 = 2$.

Solution. Suppose that $z = x + iy$ so that $z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$ and $(\overline{z})^2 = (x-iy)^2 = x^2 - y^2 - i2xy$. This implies

$$z^2 + (\overline{z})^2 = (x^2 - y^2 + i2xy) + (x^2 - y^2 - i2xy) = 2x^2 - 2y^2$$

and so the set of $z \in \mathbb{C}$ satisfying $z^2 + (\overline{z})^2 = 2$ is equivalent to the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$$

which describes a hyperbola as shown in Figure 3.5.

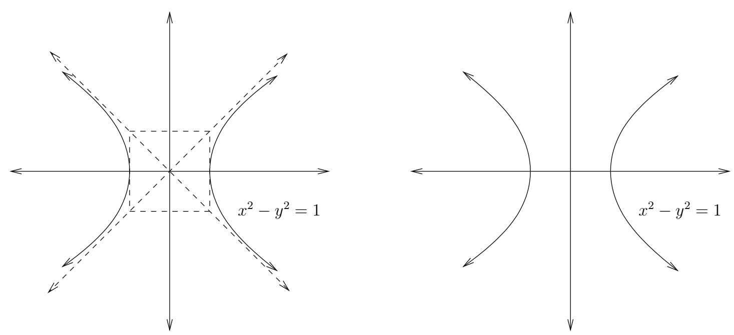

Figure 3.5: The hyperbola $x^2 - y^2 = 1$.

Exercise 3.6. Describe the curve generated by $|z+3| + |z-3| = 10$.

Solution. The curve is an ellipse which can be described in cartesian coordinates by

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1.$$

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Lecture #4: Polar Form of a Complex Variable

Suppose that $z = x + iy$ is a complex variable. Our goal is to define the polar form of a complex variable. We start with the definition. We will then describe how our experience with real variables motivates this definition.

Definition. Suppose that $z = x + iy \in \mathbb{C}$ with $z \neq 0$. The *polar form* of z is defined as $re^{i\theta}$ where $r \geq 0$ satisfies $r = \sqrt{x^2 + y^2}$ and θ is the unique angle in $(-\pi, \pi]$ satisfying

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

The polar form of $z = 0$ is, by convention, $z = 0e^{i0} = 0$.

Consider the pair $(x, y) \in \mathbb{R}^2$ in cartesian coordinates. We know from Math 213 that an equivalent way to describe a point in the plane is in terms of polar coordinates. That is, we can describe the point (x, y) in terms of its distance r from the origin and the angle the point makes with the positive x-axis. This leads to the change-of-variables

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$. If we try to invert this transformation and solve for r and θ, then we find

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(y/x).$$

The trouble here is that the inverse equation

$$\theta = \arctan(y/x)$$

is not true for pairs (x, y) in the second or third quadrants. The reason for this is the *convention* that the standard interpretation of the arctangent function places its range in the first and fourth quadrants; that is, by convention, the domain of the tangent function is restricted to $(-\pi/2, \pi/2)$ in order for the inverse of tangent function to be single-valued. Note the reason for this convention. The only asymptotes of the tangent function on $(-\pi/2, \pi/2)$ are at the endpoints. If instead we considered the tangent function on the interval $[0, \pi]$, then we would have the issue that the tangent function is not defined at $\pi/2$. This would then lead to the domain of the tangent function being $[0, \pi/2) \cup (\pi/2, \pi]$ which is ugly. The conclusion is that we cannot define θ simply as $\theta = \arctan(y/x)$ and so we instead define it as the *unique* angle $\theta \in (-\pi, \pi]$ satisfying

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

Question. When we are working with real variables (in particular in Math 213), we define θ as the *unique* angle $\theta \in [0, 2\pi)$ satisfying

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

In the definition of the polar form of a complex variable, why don't we define θ in the same way?

Answer. While it is true that there is a unique angle θ in *any* half-open half-closed interval of length 2π satisfying

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|},$$

we must make a *convention* as to which choice of definite interval we wish to make. We declare $(-\pi, \pi]$ as our convention.

Perhaps you will like this better.

Definition. Suppose that $z = x + iy \in \mathbb{C}$, $z \neq 0$. Define the *argument* of z, denoted $\arg z$, to be *any* solution θ of the pair of equations

$$\cos\theta = \frac{x}{|z|} \quad \text{and} \quad \sin\theta = \frac{y}{|z|}.$$

Note that if θ_0 qualifies as a value of $\arg z$, then so do

$$\theta_0 \pm 2\pi, \theta_0 \pm 4\pi, \theta_0 \pm 6\pi, \ldots.$$

Moreover, every value of $\arg z$ must be one of these.

However, we still have the problem of multi-valuedness. For definiteness, we will want only a single value of the argument. This leads to the following definition.

Definition. Suppose that $z = x + iy \in \mathbb{C}$. Define the *principal value of the argument* of z, denoted $\operatorname{Arg} z$, to be the unique value of $\arg z \in (-\pi, \pi]$.

Note that we did not avoid the convention that the angle belong to $(-\pi, \pi]$. We just hid it in our definitions. Actually, there is a more sophisticated reason for the convention that $\operatorname{Arg} z \in (-\pi, \pi]$. This has to do with the definition of square root. We will want to maintain the convention that the square root of a positive real number is a positive real number. This is easiest to achieve if we choose $\operatorname{Arg} z \in (-\pi, \pi]$.

Definition. Suppose that $z \in \mathbb{C}$. We define the *polar form* of z to be $re^{i\theta}$ where $r = |z|$ and $\theta = \operatorname{Arg} z$. For convenience, we will write $z = re^{i\theta}$.

Note that if $z = 0$, then we take, by convention, $\operatorname{Arg} 0 = 0$ and $\arg 0 = \{0, \pm 2\pi, \pm 4\pi, \ldots\}$.

Example 4.1. Write $z = 1 + i$ in polar form and identify $\arg z$.

Solution. If $z = 1 + i$, then

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2} = r.$$

Moreover,

$$\cos\theta = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{2}}$$

implies that

$$\theta = \frac{\pi}{4} \pm 2\pi k$$

for $k \in \mathbb{Z}$. Thus,

$$\operatorname{Arg} z = \frac{\pi}{4} \quad \text{and} \quad \arg z = \left\{\frac{\pi}{4}, \frac{\pi}{4} \pm 2\pi, \frac{\pi}{4} \pm 4\pi, \ldots\right\} = \left\{\frac{\pi}{4} \pm 2\pi k : k \in \mathbb{Z}\right\}.$$

Hence, the polar form of $z = 1 + i$ is

$$\sqrt{2}e^{i\pi/4}.$$

Equivalently, we can represent z as an ordered pair $(x, y) \in \mathbb{R}^2$ as

$$z = (1, 1) = (\sqrt{2}\cos(\pi/4), \sqrt{2}\sin(\pi/4)).$$

Suppose that $z = re^{i\theta}$ is the polar form of $z \in \mathbb{C}$. As in the previous example, we can write z in cartesian coordinates as $z = (r\cos\theta, r\sin\theta)$. Using our identification of $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$, we conclude that an equivalent representation of z is

$$z = r\cos\theta + ir\sin\theta.$$

This is sometimes called the polar form of z.

Definition. Suppose that $z \in \mathbb{C}$. The *polar form* of z is defined as

$$z = (r\cos\theta, r\sin\theta) = r\cos\theta + ir\sin\theta = re^{i\theta}$$

where $r = |z|$ and $\theta = \operatorname{Arg} z$.

If we take $r = 1$ in the definition of polar form, then we conclude that

$$(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta = e^{i\theta}$$

which leads to the following definition.

Definition. The *complex exponential* $e^{i\theta}$ is defined as $e^{i\theta} = \cos\theta + i\sin\theta$.

Properties of the Complex Exponential $e^{i\theta}$

Proposition 4.2. $e^{-i\theta} = \overline{e^{i\theta}}$

Proof. We find

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta) = \overline{e^{i\theta}}$$

and the proof is complete. □

Proposition 4.3. $|e^{i\theta}| = 1$

Proof. Using the previous proposition, we find

$$|e^{i\theta}| = e^{i\theta}\overline{e^{i\theta}} = e^{i\theta}e^{-i\theta} = (\cos(\theta) + i\sin(\theta))(\cos(\theta) - i\sin(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

as required. $\square$

Proposition 4.4. $\dfrac{1}{e^{i\theta}} = e^{-i\theta}$

Proof. We find

$$\begin{aligned}\frac{1}{e^{i\theta}} = \frac{1}{\cos(\theta) + i\sin(\theta)} = \frac{1}{\cos(\theta) + i\sin(\theta)}\frac{\cos(\theta) - i\sin(\theta)}{\cos(\theta) - i\sin(\theta)} &= \frac{\cos(\theta) - i\sin(\theta)}{|e^{i\theta}|} \\ &= \cos(\theta) - i\sin(\theta) \\ &= e^{-i\theta}\end{aligned}$$

and the proof is complete. $\square$

Proposition 4.5. $e^{i\theta} = e^{i(\theta+2\pi k)}$, $k \in \mathbb{Z}$

Proof. Since the real-valued sine and cosine functions are each 2π-periodic, we know that

$$\cos(\theta) = \cos(\theta + 2\pi k) \quad \text{and} \quad \sin(\theta) = \sin(\theta + 2\pi k)$$

so that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) = \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) = e^{i(\theta+2\pi k)}$$

as required. $\square$

Proposition 4.6. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Proof. By definition,

$$\begin{aligned}e^{i\theta_1}e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \\ &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \\ &= e^{i(\theta_1+\theta_2)}\end{aligned}$$

completing the proof. $\square$

Proposition 4.7. $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$

Proof. Using our previous propositions, we find

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i\theta_1}e^{-i\theta_2} = e^{i\theta_1 - i\theta_2} = e^{i(\theta_1-\theta_2)}$$

as required. □

Corollary 4.8. *If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

and if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Exercise 4.9. Prove the previous corollary.

Powers: An Application of Complex Exponentials

Recall that if $a \in \mathbb{R}$ and n, $m \in \mathbb{Z}$, then $(a^n)^m = a^{nm}$. In particular, if $x \in \mathbb{R}$, then $(e^x)^n = e^{nx}$. As we will now show, this same sort of result is true for the complex exponential.

Theorem 4.10. *Let $z = re^{i\theta}$ be the polar form of the complex variable z. If n is a non-negative integer, then*

$$z^n = r^n e^{in\theta}.$$

Proof. The proof is by induction. Clearly it is true for $n = 1$. If $n = 2$, then we find from Corollary 4.8 that

$$z^2 = (re^{i\theta})(re^{i\theta}) = r^2 e^{i(\theta+\theta)} = r^2 e^{i2\theta}.$$

If $n = 3$, then

$$z^3 = z^2 z = (r^2 e^{i2\theta})(re^{i\theta}) = r^3 e^{i(2\theta+\theta)} = r^3 e^{i3\theta}.$$

In general, if $z^k = r^k e^{ik\theta}$ for some k, then

$$z^{k+1} = z^k z = (r^k e^{ik\theta})(re^{i\theta}) = r^{k+1} e^{i(k\theta+\theta)} = r^{k+1} e^{i(k+1)\theta}$$

which completes the proof. □

We can now use this theorem to derive de Moivre's formula.

Theorem 4.11 (de Moivre's Formula)**.** *If n is a positive integer, then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. Consider $z = \cos\theta + i\sin\theta$ so that the polar form of z is $z = e^{i\theta}$. On the one hand we have

$$z^n = (\cos\theta + i\sin\theta)^n.$$

On the other hand we have

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

Equating the two gives

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

as required. $\square$

Remark. If we take $\theta = \pi$ (and $r = 1$) in definition of complex exponential, then we have one of the most magical formulas in all of mathematics:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1,$$

or equivalently,

$$\boxed{e^{i\pi} + 1 = 0}$$

which is a formula relating all five fundamental constants of mathematics!!!! The constant e comes from calculus, π comes from geometry, i comes from algebra, and 1 is the basic unit for generating the arithmetic system from the usual counting numbers.

Lecture #5: Applications of Complex Exponentials

Recall from last class that we defined the complex exponential $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Using this we concluded that the polar form of $z \in \mathbb{C}$ can be written as

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$$

where $r = |z|$ and $\theta = \operatorname{Arg}(z)$. We also derived de Movire's formula, namely

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for any positive integer n.

Remark. On numerous occasions we have seen that the motivation for a complex variables definition comes from the corresponding real variable definition. Therefore, it is natural to ask whether the definition of $e^{i\theta}$ is consistent with the definition from calculus. Recall that the Taylor series for e^x about $x = 0$ is

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Therefore, if we take $x = i\theta$, we have

$$\begin{aligned} e^{i\theta} = \sum_{j=0}^{\infty} \frac{(i\theta)^j}{j!} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

We now observe that

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

If we solve this system of equations for $\cos\theta$ and $\sin\theta$, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Definition. If $z = x + iy \in \mathbb{C}$, we define the *complex exponential* e^z as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Note that

$$|e^z| = |e^x e^{iy}| = |e^x||e^{iy}| = |e^x| = e^x$$

since $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$ and $e^x > 0$ for $x \in \mathbb{R}$. In particular, if $\mathrm{Re}(z) \leq 0$, then $|e^z| \leq 1$.

Example 5.1. Find an identity for

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) \tag{$*$}$$

where n is a positive integer and $\theta \in \mathbb{R}$. Note that in the study of Fourier series it is important to be able to evaluate such an expression.

Before solving this problem, we need to establish a preliminary result. Recall the formula for a geometric series. If $x \in \mathbb{R}$ with $x \neq 1$, then

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

for any positive integer n. Moreover, if $|x| < 1$, then we can let $n \to \infty$ to obtain

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

Proposition 5.2. *If $z \in \mathbb{C}$ with $z \neq 1$, then*

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \tag{$**$}$$

for any positive integer n.

Proof. Since

$$(1 + z + z^2 + \cdots + z^n)(1 - z) = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}$$

and $z \neq 1$ we can divide by $(1 - z)$ to obtain the result. $\square$

Solution. We can now find an identity for $(*)$. If we take $z = e^{i\theta}$ in $(**)$, we obtain

$$1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^n = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}}$$

or, equivalently,

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Taking the real parts of the previous express implies that

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) = \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right).$$

We will now find a simple expression for the right side of the previous equality. Note that

$$\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\theta/2}}{e^{i\theta/2}-e^{i\theta/2}} = \frac{1}{2i}\frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\theta/2}}{\sin(\theta/2)}.$$

Now observe that

$$\begin{aligned} e^{i(n+\frac{1}{2})\theta}-e^{-i\theta/2} &= \left[\cos((n+\tfrac{1}{2})\theta)+i\sin((n+\tfrac{1}{2})\theta)\right]-\left[\cos(\theta/2)-i\sin(\theta/2)\right] \\ &= \cos((n+\tfrac{1}{2})\theta)-\cos(\theta/2)+i\left[\sin((n+\tfrac{1}{2})\theta)+\sin(\theta/2)\right] \end{aligned}$$

and so

$$\begin{aligned} \mathrm{Re}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right) &= \mathrm{Re}\left[\frac{1}{2i}\frac{e^{i(n+\frac{1}{2})\theta}-e^{-i\theta/2}}{\sin(\theta/2)}\right] \\ &= \frac{1}{2\sin(\theta/2)}\,\mathrm{Re}\left[\tfrac{1}{i}\left(\cos((n+\tfrac{1}{2})\theta)-\cos(\theta/2)+i\left[\sin((n+\tfrac{1}{2})\theta)+\sin(\theta/2)\right]\right)\right] \\ &= \frac{1}{2\sin(\theta/2)}\left[\sin((n+\tfrac{1}{2})\theta)+\sin(\theta/2)\right]. \end{aligned}$$

That is,

$$1+\cos\theta+\cos(2\theta)+\cdots+\cos(n\theta) = \frac{\sin((n+\frac{1}{2})\theta)+\sin(\theta/2)}{2\sin(\theta/2)}.$$

Example 5.3. Express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin(3\theta)$.

Solution. We know from de Moivre's formula that $(\cos\theta+i\sin\theta)^n = \cos(n\theta)+i\sin(n\theta)$ for any positive integer n and so

$$\sin(3\theta) = \mathrm{Im}[(\cos\theta+i\sin\theta)^3].$$

We know from the binomial theorem that

$$(a+b)^n = \sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}$$

and so

$$(x+iy)^3 = x^3+3x^2(iy)+3x(iy)^2+(iy)^3 = x^3-3xy^2+i(3x^2y-y^3).$$

Taking $x=\cos\theta$ and $y=\sin\theta$ yields

$$(\cos\theta+i\sin\theta)^3 = \cos^3\theta-3\cos\theta\sin^2\theta+i(3\cos^2\theta\sin\theta-\sin^3\theta)$$

which in turn implies that

$$\sin(3\theta) = 3\cos^2\theta\sin\theta-\sin^3\theta.$$

Substituting in $\sin^2\theta+\cos^2\theta=1$ gives

$$\sin(3\theta) = 3(1-\sin^2\theta)\sin\theta-\sin^3\theta = 3\sin\theta-3\sin^3\theta-\sin^3\theta = 3\sin\theta-4\sin^3\theta$$

so that

$$\sin^3\theta = \frac{3}{4}\sin\theta-\frac{1}{4}\sin(3\theta).$$

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Lecture #6: Powers and Roots of Algebraic Equations

Recall from high school that one of the things we do with real numbers is solve equations for them. For instance, we can ask for all values of $x \in \mathbb{R}$ such that $2x+3=0$. The answer, of course, is $x=-3/2$. We can then try to solve more sophisticated equations. For instance, we can ask for all values of $x \in \mathbb{R}$ such that $x^3-2x^2-x=0$. There are three possible values of x, namely

$$x \in \{0, 1-\sqrt{2}, 1+\sqrt{2}\}.$$

However, we quickly discover that not every equation has a real solution. For instance, there are no real values of x such that $x^2+1=0$. Indeed, this is one of the motivations for introducing complex variables. Having completed our study of the arithmetic of complex variables, we will now start to solve equations involving them. We will then discover the Fundamental Theorem of Algebra which will tell us that any polynomial of degree n will have n complex roots.

Example 6.1. Let $a \neq 0, b, c \in \mathbb{C}$ be given. Find all values of $z \in \mathbb{C}$ such that $az^2+bz+c=0$.

Solution. Just as in high school we can use the quadratic formula. That is,

$$az^2+bz+c = a\left(z^2+\frac{bz}{a}\right)+c = a\left(z^2+\frac{bz}{a}+\frac{b^2}{4a}\right)-\frac{b^2}{4}+c = a\left(z+\frac{b}{2a}\right)^2-\frac{b^2}{4}+c.$$

We would now like to take square roots to obtain two solutions, namely

$$z = \sqrt{\frac{b^2}{4a}-\frac{c}{a}}-\frac{b}{2a} = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

and

$$z = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

It is natural to ask, however, if this is truly a valid solution; in particular, what is meant by a complex square root.

nth Roots of a Complex Variable

Instead of restricting ourselves to just square roots of complex variables, we will consider nth roots. Suppose that $z \in \mathbb{C}$ and that n is a positive integer. Our goal is to determine $\sqrt[n]{z} = z^{1/n}$. By definition, this means that we must find all values of $\zeta \in \mathbb{C}$ such that $\zeta^n = z$. Suppose that we write $\zeta = \rho e^{i\varphi}$ where $\rho = |\zeta|$ and $\varphi = \mathrm{Arg}(\zeta)$. Suppose further that $\rho = 1$ so that ζ lies on the unit circle. Therefore,

$$\zeta^2 = e^{i2\varphi} \quad \text{and} \quad \zeta^3 = e^{i3\varphi}$$

also lie on the unit circle as illustrated in Figure 6.1.

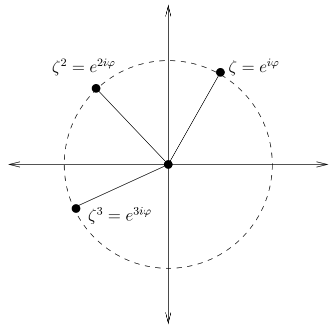

Figure 6.1: Geometric interpretation of $\zeta = e^{i\varphi}$, ζ^2, ζ^3.

Indeed, it is clear that when $|\zeta| = 1$, successive powers of ζ (which represent repeated multiplication by ζ) correspond to successive rotations by an angle of φ in the complex plane. This important idea will be expanded upon later.

Example 6.2. Find the two square roots of 1; that is, determine all values of $z \in \mathbb{C}$ such that $z^2 = 1$.

Solution. It is pretty clear that the two solutions are $\zeta_1 = 1$ and $\zeta_2 = -1$. However, we will derive these solutions using the polar form of a complex variable as this is the method that will work in more generality. Therefore, suppose that $\zeta = e^{i\varphi}$ so that $|\zeta| = 1$. We also write 1 as $1 = e^{i0}$. In order to have $\zeta^2 = 1$ we must have

$$\zeta^2 = (e^{i\varphi})^2 = e^{i2\varphi} = e^{i0} = 1.$$

This implies that $2\varphi = 0$ so $\varphi = 0$. The result is

$$\zeta = e^{i\varphi} = e^{i0} = 1.$$

However, we must also realize that there is more than one value of θ for which $e^{i\theta} = 1$. Indeed, $e^{i2\pi} = 1$. Therefore, in order to have $\zeta^2 = 1$, we must have

$$\zeta^2 = (e^{i\varphi})^2 = e^{i2\varphi} = e^{i2\pi} = 1.$$

This implies that $2\varphi = 2\pi$ so $\varphi = \pi$. The result is

$$\zeta = e^{i\varphi} = e^{i\pi} = -1.$$

Of course, the issue is the fact that $e^{i\theta}$ is multivalued. Indeed,

$$e^{i\theta} = e^{i(\theta + 2\pi k)}, \quad k \in \mathbb{Z}.$$

This means in order to find the two solutions to $z^2 = 1$, we must find the two values of $\varphi \in [0, 2\pi)$ for which $e^{i2\varphi} = 1$. The answer, as we discovered, is $\varphi \in \{0, \pi\}$.

Example 6.3. Find the three cube roots of 1; that is, determine all values of $z \in \mathbb{C}$ such that $z^3 = 1$.

Solution. Suppose that $\zeta = e^{i\varphi}$. We need to find the three values of $\varphi \in [0, 2\pi)$ such that

$$e^{i3\varphi} = 1.$$

If we write $1 = e^{i0}$, then the first value is $\varphi_1 = 0$ since

$$e^{i3\varphi_1} = e^{i0}.$$

If we write $1 = e^{i2\pi}$, then the second value is $\varphi_2 = 2\pi/3$ since

$$e^{i3\varphi_2} = e^{i2\pi}.$$

If we write $1 = e^{i4\pi}$, then the third value is $\varphi_3 = 4\pi/3$ since

$$e^{i3\varphi_3} = e^{i2\pi}.$$

Thus, the three solutions are

$$\zeta_1 = 1, \quad \zeta_2 = e^{i2\pi/3}, \quad \zeta_3 = e^{i4\pi/3}.$$

Note that you might object since $4\pi/3$ does not lie in the interval $(-\pi, \pi]$ which means that $\mathrm{Arg}(\zeta_3) = \mathrm{Arg}(e^{i4\pi/3})$ does not equal $4\pi/3$. Since $\mathrm{Arg}(e^{i4\pi/3}) = -2\pi/3$, you might prefer to write

$$\zeta_3 = e^{-i2\pi/3}$$

instead. You can also find this solution as follows. If we write $1 = e^{-i2\pi}$, then the third value is $\varphi_3 = -2\pi/3$ since

$$e^{i3\varphi_3} = e^{-i2\pi}.$$

It is important to stress that in cartesian coordinates there is no ambiguity. The three cube roots of 1 are

$$\zeta_1 = 1, \quad \zeta_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \zeta_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Exercise 6.4. Determine all values of $z \in \mathbb{C}$ such that $z^5 = 1$.

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Lecture #7: Powers and Roots of Algebraic Equations

Example 7.1. Find all values of $z \in \mathbb{C}$ such that $z^4 = 1$.

Solution. Suppose that ζ is a solution to the equation. We begin by noting that $\zeta^4 = 1$ implies $|\zeta|^4 = 1$ which in turn implies $|\zeta| = 1$ so that ζ lies on the unit circle. Therefore, we assume that the polar form of ζ is $\zeta = e^{i\varphi}$ and so we need to solve

$$\zeta^4 = e^{i4\varphi} = e^{i0} = 1.$$

However, we know that

$$e^{i\varphi} = e^{i(\varphi+2k\pi)} \quad \text{for} \quad k \in \mathbb{Z}.$$

Since we want $\varphi \in [0, 2\pi)$, we conclude that

$$4\varphi \in \{0, 2\pi, 4\pi, 6\pi\} \quad \text{so that} \quad \varphi \in \{0, \pi/2, \pi, 3\pi/2\}.$$

Thus, there are four solutions to $z^4 = 1$, namely

$$\zeta_1 = e^{i0} = 1, \quad \zeta_2 = e^{i\pi/2} = i, \quad \zeta_3 = e^{i\pi} = -1, \quad \zeta_4 = e^{i3\pi/2} = -i.$$

We can plot these solutions in the complex plane.

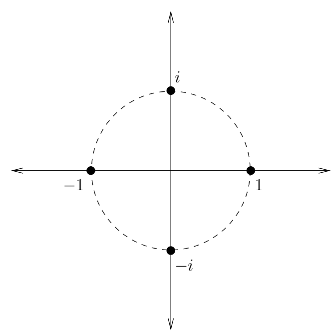

Figure 7.1: Geometric representation of solutions to $z^4 = 1$.

Also note that $\zeta_j^4 = 1$ for each $j = 1, 2, 3, 4$. Therefore, since multiplication of complex variables of unit modulus corresponds to rotation, we can conclude that the four roots are related to each other by a rotation of $2\pi/4 = \pi/2$ radians.

We can now generalize the previous example.

Example 7.2. Find all values of $z \in \mathbb{C}$ such that $z^n = 1$ where n is a positive integer.

Solution. Note that any solution will necessarily have modulus 1. Therefore, consider $\zeta = e^{i\varphi}$. There are n values of φ in $[0, 2\pi)$ for which

$$\zeta^n = e^{in\varphi} = 1$$

holds.

In fact, they satisfy

$$n\varphi = 2k\pi, \quad k = 0, 1, \ldots, n-1$$

or, equivalently,

$$\varphi = \frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

Thus, the n solutions are

$$\zeta_1 = e^{i0} = 1, \quad \zeta_2 = e^{i2\pi/n}, \quad \zeta_3 = e^{i4\pi/n}, \quad \ldots, \quad \zeta_n = e^{i2(n-1)\pi/n}.$$

We call

$$\{\zeta_1 = 1, \zeta_2, \ldots, \zeta_n\}$$

the n *roots of unity.*

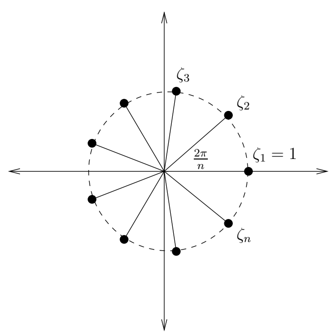

Figure 7.2: Geometric representation of the n roots of unity.

We can represent the n roots of unity as n points equally spaces around the circle of radius 1. Note that $\zeta_1 = 1$, and each subsequent root is obtained by rotating the previous root by $2\pi/n$ radians. After n rotations, we are back to our starting point.

The Roots of Unity

We will now consider a different notation for the roots of unity. Let

$$\omega_n = e^{i2\pi/n}.$$

Note that

$$\omega_n^0 = 1, \quad \omega_n^2 = (e^{i2\pi/n})^2 = e^{i4\pi/n}, \ldots, \omega_n^k = (e^{i2\pi/n})^k = e^{i2\pi k/n}, \ldots$$

and so the n roots of unity can be written as

$$\{\omega_n^0 = 1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}.$$

The geometric interpretation to the roots of unity that we gave above is perhaps even more clearly illustrated with this notation. The first root of unity is $\omega_n^0 = 1$. The second root of unity is $\omega_n = \omega_n \cdot 1$ which represents rotation by $2\pi/n$ degrees from 1. The third root of unity is $\omega_n^2 = \omega_n \cdot \omega_n$ which represents rotation by $2\pi/n$ degrees from $e^{i2\pi/n}$. In general, subsequent roots of unity can be obtained from the previous root by rotation through $2\pi/n$ radians.

Proposition 7.3. *If* $\omega_n = e^{i2\pi/n}$*, then*

$$1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = 0.$$

Proof. This follows immediately from Proposition 5.2. That is,

$$1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} = \frac{1 - \omega_n^n}{1 - \omega_n} = 0$$

since $\omega_n^n = (e^{i2\pi/n})^n = e^{i2\pi} = 1$. $\square$

Example 7.4. Determine all complex values of $(16)^{1/4}$.

Solution. We know that the fourth roots of unity are 1, -1, i, and $-i$. We know that the fourth root of the positive real number $|16|$ is 2. Thus, the possible fourth roots of the complex variable 16 are

$$\{2, -2, 2i, -2i\}.$$

Observe that we can write

$$(16)^{1/4} = |16|^{1/4} 1^{1/4}.$$

Here we are viewing $(16)^{1/4} \in \mathbb{C}$, $|16|^{1/4} \in \mathbb{R}$, and $1^{1/4} \in \mathbb{C}$.

In fact, the idea of the previous example holds in general.

Example 7.5. Suppose that $w \in \mathbb{C}$ is given. Determine all values of $z \in \mathbb{C}$ such that $z^n = w$.

Solution. Suppose that ζ is one such solution so that $\zeta^n = w$. If we write $w = re^{i\theta}$ and $\zeta = \rho e^{i\varphi}$, then we must have

$$\rho^n e^{in\varphi} = re^{i\theta}.$$

Since both ρ and r are non-negative real numbers, we must have $\rho = r^{1/n}$. Here we are writing ρ for the unique positive real valued nth root of r. Moreover, since

$$\zeta = r^{1/n} e^{i\varphi} \quad \text{so that} \quad |\zeta| = r^{1/n}$$

we see that the solutions lie on the circle of radius $r^{1/n}$. Furthermore, we know that there are n values of $\varphi \in [0, 2\pi)$ for which

$$e^{in\varphi} = e^{i\theta},$$

namely

$$n\varphi = \theta + 2k\pi, \quad k = 0, 1, \ldots, n-1,$$

or, equivalently,

$$\varphi = \frac{\theta + 2k\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

Thus, the n solutions to $z^n = w = re^{i\theta}$ are $\zeta_1, \ldots, \zeta_n$ where

$$\zeta_k = r^{1/n} e^{i(\theta + 2k\pi)/n}, \quad k = 0, 1, \ldots, n-1.$$

Lecture #8:

Mathematics 312 (Fall 2012) September 24, 2012
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Lecture #9:

Lecture #10:

Mathematics 312 (Fall 2012) September 28, 2012
Prof. Michael Kozdron

Lecture #11: Limits, Continuity, and Differentiability

Definition. Let $f(z)$ be a function defined in some neighbourhood of z_0, except possibly at z_0 itself. We say that $f(z)$ *converges to* w_0 *as* z *converges to* z_0, written

$$\lim_{z \to z_0} f(z) = w_0,$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

Definition. We say that $f(z)$ is *continuous at* z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Remark. This is the same definition as in calculus except that the condition $0 < |z - z_0| < \delta$ allows z to approach z_0 in any direction as shown in Figure 11.1. This makes limits much more subtle with complex variables.

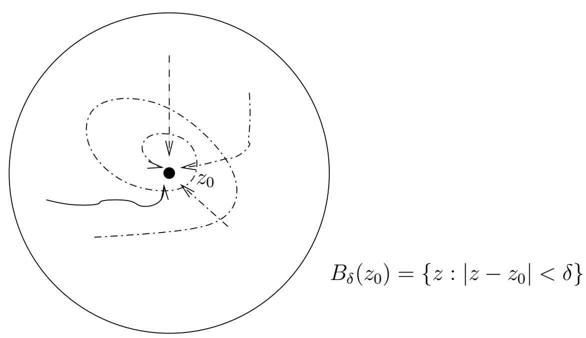

Figure 11.1: z can approach z_0 from any direction.

Definition. Let $f(z)$ be defined in a neighbourhood of z_0. The *derivative of* $f(z)$ *at* z_0 is

$$\left.\frac{\mathrm{d}}{\mathrm{d}z} f(z)\right|_{z_0} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided that the limit exists.

Remark. The limit must be independent of path $\Delta z \to 0$ in order for the derivative to exist.

Example 11.1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z$. Show that $f(z)$ is differentiable at z_0 for every $z_0 \in \mathbb{C}$.

Solution. Since

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z) - (z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

for all $z_0 \in \mathbb{C}$, we conclude that f is differentiable at z_0 for every $z_0 \in \mathbb{C}$ with $f'(z_0) = 1$.

Example 11.2. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \overline{z}$. Is $f(z)$ differentiable at $z_0 \in \mathbb{C}$?

Solution. Observe that

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \overline{(z_0)}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z},$$

and so the question is to determine what happens as $\Delta z \to 0$. In particular, is the value independent of path? To see that it is not, let $\Delta z = \Delta x + i\Delta y$ so that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Consider approaching 0 along the positive real axis. This means that $\Delta y = 0$ so that $\Delta z = \Delta x$ and

$$\Delta z \to 0 \quad \text{if and only if} \quad \Delta x \to 0.$$

Therefore, we conclude

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0,\, \Delta y = 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Now consider approaching 0 along the positive imaginary axis. This means that $\Delta x = 0$ so that $\Delta z = i\Delta y$ and

$$\Delta z \to 0 \quad \text{if and only if} \quad \Delta y \to 0.$$

Therefore, we conclude

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x = 0,\, \Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Since the value of the limit is not independent of path, we conclude that $f(z) = \overline{z}$ is nowhere differentiable!

Example 11.3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = |z|^2$. Is $f(z)$ differentiable at $z_0 \in \mathbb{C}$?

Solution. Observe that

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} &= \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0\overline{z_0}}{\Delta z} \\
&= \frac{z_0\overline{z_0} + \overline{z_0}\Delta z + z_0\overline{\Delta z} + \overline{\Delta z}\Delta z - z_0\overline{z_0}}{\Delta z} \\
&= \overline{z_0} + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}
\end{aligned}$$

and so the question is to determine what happens as $\Delta z \to 0$. Consider

$$\lim_{\Delta z \to 0} \left(\overline{z_0} + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right). \tag{$*$}$$

We know that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} \quad \text{does not exist}$$

and so the middle term in $(*)$ does not exist *except* when $z_0 = 0$. However, if $z_0 = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

This means that $f(z) = |z|^2$ is differentiable at $z_0 = 0$ with $f'(0) = 0$, but is not differentiable at any $z_0 \in \mathbb{C} \setminus \{0\}$.

Remark. The function $f(z) = \overline{z}$ is nowhere differentiable, and the function $f(z) = |z|^2$ is differentiable only at 0. As we will see more formally later, functions that involve $\overline{z}$ are typically not differentiable.

Definition. A function $f(z)$ is *analytic* in some domain D if it is differentiable at each point in D. (Recall that a domain is an open, connected set. In particular, D cannot be a single point.)

Definition. A function $f(z)$ is *analytic at* z_0 if it is differentiable at z_0 and if it is differentiable at all z in some neighbourhood of z_0.

Example 11.4. The function $f(z) = z$ is analytic at 0 since it is differentiable at 0 and is differentiable at all z in any neighbourhood of 0. (In fact, $f(z) = z$ is analytic in $\mathbb{C}$.) The function $f(z) = |z|^2$ is differentiable at 0, but it is not analytic at 0 since it is not differentiable at any $z \neq 0$.

Remark. The usual properties of limits that we had in calculus for real functions also hold for complex function. In particular, we can deduce the following results from their real variable counterparts:

- $\dfrac{\mathrm{d}}{\mathrm{d}z}(f(z)g(z)) = f'(z)g(z) + g'(z)f(z)$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{f'(z)g(z) - g'(z)f(z)}{g^2(z)}$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z}(f \circ g)(z) = f'(g(z))g'(z)$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z}(\alpha f(z) + \beta) = \alpha f'(z) + \beta$, for any α, $\beta \in \mathbb{C}$.

Note. When answering the problems on Assignment #3, note that the only derivative results we have proved are the following.

- If $f(z) = z$, then $f'(z) = 1$.
- If $f(z) = \overline{z}$, then $f'(z)$ does not exist for any $z \in \mathbb{C}$.
- If $f(z) = |z|^2$, then $f'(0) = 0$ and $f'(z)$ does not exist for any $z \in \mathbb{C} \setminus \{0\}$.

For any other complex function $f(z)$, you need to use the definition of derivative to determine $f'(z)$.

Mathematics 312 (Fall 2012) October 1, 2012
Prof. Michael Kozdron

Lecture #12: Selected Review of Assignments #2 and #3

Example 12.1 (Assignment #2, Problem #10)**.** Solve the equation $(z+1)^5 = z^5$ for $z \in \mathbb{C}$.

Solution. Consider the function $f(z) = (z+1)^5 - z^5$. If we expand $(z+1)^5$ we see that the leading term is z^5. Hence, $f(z)$ is a *fourth* degree polynomial. Thus, there are four complex variables ζ_1, ζ_2, ζ_3, ζ_4 with $f(\zeta_j) = 0$. Observe that $(z+1)^5 = z^5$ is equivalent to

$$1 = \frac{(z+1)^5}{z^5} = \left(\frac{z+1}{z}\right)^5 = \left(1 + \frac{1}{z}\right)^5$$

for all $z \in \mathbb{C} \setminus \{0\}$. If we let $w = 1 + z^{-1}$, then $w^5 = 1$ implies that

$$w \in \{1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}\}$$

and so

$$1 + z^{-1} \in \{1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}\}.$$

Since $z = 0$ is clearly not a solution to $(z+1)^5 = z^5$, we conclude that $1 + z^{-1} = 1$ is not a legitimate solution to $(1 + z^{-1})^5 = 1$. (That is, $1 + z^{-1} = 1$ implies $z^{-1} = 0$ which is nonsensical.) Hence, $1 + z^{-1} \in \{e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}\}$ so that

$$\zeta_1 = \frac{1}{e^{i2\pi/5} - 1}, \quad \zeta_2 = \frac{1}{e^{i4\pi/5} - 1}, \quad \zeta_3 = \frac{1}{e^{i6\pi/5} - 1}, \quad \zeta_4 = \frac{1}{e^{i8\pi/5} - 1}.$$

Example 12.2 (Assignment #2, Problem #3)**.** Use de Moivre's formula to prove that

$$\sin(2\theta) + \cdots + \sin(2n\theta) = \frac{\sin(n\theta)\sin((n+1)\theta)}{\sin(\theta)}$$

for $0 < \theta < \pi$.

Solution. Recall that if $z \neq 1 \in \mathbb{C}$, then

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Hence, if we take $z = e^{i2\theta}$ with $0 < \theta < \pi$, then

$$1 + e^{i2\theta} + (e^{i2\theta})^2 + \cdots + (e^{i2\theta})^n = \frac{1 - (e^{i2\theta})^{n+1}}{1 - e^{i2\theta}}$$

so that de Moivre's formula implies

$$1 + \cos(2\theta) + \cdots + \cos(2n\theta) + i[\sin(2\theta) + \cdots + \sin(2n\theta)] = \frac{1 - e^{i2(n+1)\theta}}{1 - e^{i2\theta}}.$$

We will write the right side of the previous equation as

$$\frac{1-e^{i2(n+1)\theta}}{1-e^{i2\theta}} = \frac{1-e^{i2\theta}+e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}} = 1 + \frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}}$$

so that

$$\cos(2\theta)+\cdots+\cos(2n\theta)+i[\sin(2\theta)+\cdots+\sin(2n\theta)] = \frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}}.$$

Thus, we conclude that

$$\sin(2\theta)+\cdots+\sin(2n\theta) = \operatorname{Im}\left[\frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}}\right].$$

Multiplying and dividing by $e^{-i\theta}$ implies

$$\frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}} = \frac{e^{-i\theta}}{e^{-i\theta}}\frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}} = \frac{e^{i\theta}-e^{i(2n+1)\theta}}{e^{-i\theta}-e^{i\theta}}.$$

We now write

$$e^{i\theta}-e^{i(2n+1)\theta} = e^{i\theta}(1-e^{i2n\theta}) = e^{i\theta}e^{in\theta}(e^{-in\theta}-e^{in\theta}) = e^{i(n+1)\theta}(e^{-in\theta}-e^{in\theta})$$

and so

$$\frac{e^{i\theta}-e^{i(2n+1)\theta}}{e^{-i\theta}-e^{i\theta}} = e^{i(n+1)\theta}\frac{(e^{-in\theta}-e^{in\theta})}{e^{-i\theta}-e^{i\theta}} = e^{i(n+1)\theta}\frac{\sin(n\theta)}{\sin(\theta)}.$$

Therefore,

$$\operatorname{Im}\left[\frac{e^{i2\theta}-e^{i2(n+1)\theta}}{1-e^{i2\theta}}\right] = \operatorname{Im}\left[e^{i(n+1)\theta}\frac{\sin(n\theta)}{\sin(\theta)}\right] = \frac{\sin(n\theta)}{\sin(\theta)}\operatorname{Im}\left[e^{i(n+1)\theta}\right] = \frac{\sin(n\theta)\sin((n+1)\theta)}{\sin(\theta)}$$

so that

$$\sin(2\theta)+\cdots+\sin(2n\theta) = \frac{\sin(n\theta)\sin((n+1)\theta)}{\sin(\theta)}$$

as required.

Example 12.3 (Assignment #3, Problem #4)**.** Show that the Joukowski function defined by

$$w = J(z) = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

maps the circle $\{|z| = r, r > 0, r \neq 1\}$ onto the ellipse

$$\frac{u^2}{\left(\frac{1}{2}\left(r+\frac{1}{r}\right)\right)^2} + \frac{v^2}{\left(\frac{1}{2}\left(r-\frac{1}{r}\right)\right)^2} = 1$$

which has foci at ± 1.

Solution 1. Consider the circle $\{|z| = r, r > 0, r \neq 1\}$. Suppose that we write z in polar coordinates as $z = re^{i\theta}$ and that we write $w = J(z)$ as $w = u + iv$. Hence,

$$\begin{aligned} u + iv = w = J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) &= \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) \\ &= \frac{1}{2}\left[r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - \frac{i}{r}\sin\theta\right] \\ &= \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + \frac{i}{2}\left(r - \frac{1}{r}\right)\sin\theta \end{aligned}$$

which implies that

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta.$$

If we then solve for $\cos\theta$ and $\sin\theta$ in the previous expressions, we obtain

$$\left[\frac{u}{\frac{1}{2}\left(r + \frac{1}{r}\right)}\right]^2 + \left[\frac{v}{\frac{1}{2}\left(r - \frac{1}{r}\right)}\right]^2 = \cos^2\theta + \sin^2\theta = 1.$$

In other words, J maps the circle $\{|z| = r, r > 0, r \neq 1\}$ onto the ellipse

$$\frac{u^2}{\left(\frac{1}{2}\left(r + \frac{1}{r}\right)\right)^2} + \frac{v^2}{\left(\frac{1}{2}\left(r - \frac{1}{r}\right)\right)^2} = 1.$$

Solution 2. If we write z in cartesian coordinates as $z = x + iy$ and $w = J(z)$ as $w = u + iv$, then we obtain

$$\begin{aligned} 2(u + iv) = 2w = 2J(z) = z + \frac{1}{z} = x + iy + \frac{1}{x + iy} &= x + iy + \frac{x - iy}{x^2 + y^2} \\ &= \left(x + \frac{x}{x^2 + y^2}\right) + i\left(y - \frac{y}{x^2 + y^2}\right) \end{aligned}$$

which implies that

$$2u = x + \frac{x}{x^2 + y^2} \quad \text{and} \quad 2v = y - \frac{y}{x^2 + y^2}.$$

We know that $x^2 + y^2 = r^2$, and so we can write the previous expressions as

$$2u = x\left(1 + \frac{1}{r}\right) \quad \text{and} \quad 2v = y\left(1 - \frac{1}{r}\right).$$

Thus, if we solve the previous expressions for x and y, square them, and add them, we obtain

$$r^2 = x^2 + y^2 = \left[\frac{2u}{1 + \frac{1}{r}}\right]^2 + \left[\frac{2v}{1 - \frac{1}{r}}\right]^2$$

which, after some arithmetic, is equivalent to

$$\frac{u^2}{\left(\frac{1}{2}\left(r + \frac{1}{r}\right)\right)^2} + \frac{v^2}{\left(\frac{1}{2}\left(r - \frac{1}{r}\right)\right)^2} = 1.$$

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Prof. Michael Kozdron

Lecture #13: Analyticity and the Cauchy-Riemann Equations

Question. Suppose that $f(z) = u(z) + iv(z)$. Under what conditions on $u = u(z) = u(x, y)$ and $v = v(z) = v(x, y)$ is $f(z)$ analytic?

Answer. We certainly need f to be differentiable at z_0. This means that f is defined in some neighbourhood of z_0 and

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{$*$}$$

exists. (In particular, the value of the limit is independent of the path $\Delta z \to 0$.) Let $\Delta z = \Delta x + i\Delta y$. We know that $(*)$ exists if (i) $\Delta y = 0$ and $\Delta x \to 0$, and (ii) $\Delta x = 0$ and $\Delta y \to 0$. Consider first the case $\Delta y = 0$. We have

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\
&= \frac{f(x_0 + \Delta x + iy_0) - f(x_0 + iy_0)}{\Delta x} \\
&= \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - (u(x_0, y_0) + iv(x_0, y_0))}{\Delta x} \\
&= \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}
\end{aligned}$$

Now consider the case $\Delta x = 0$. We have

$$\begin{aligned}
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\
&= \frac{f(x_0 + iy_0 + i\Delta y) - f(x_0 + iy_0)}{i\Delta y} \\
&= \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - (u(x_0, y_0) + iv(x_0, y_0))}{i\Delta y} \\
&= \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}
\end{aligned}$$

Since both of these are expressions for $f'(z_0)$ in the limit, we obtain by equating real and imaginary parts that

$$\lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}$$

and

$$\lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = -\lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}.$$

Equivalently,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

These are the celebrated *Cauchy-Riemann equations.*

Theorem 13.1. *If $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, then the Cauchy-Riemann equations are satisfied at $z_0 = x_0 + iy_0$; that is, if $f'(z_0)$ exists, then*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \textit{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

This theorem is most useful, however, when considered in the contrapositive.

Corollary 13.2. *Consider $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$. If the Cauchy-Riemann equations are not satisfied by f at (x_0, y_0), then f is not differentiable at z_0. In particular, if f is not differentiable at z_0, then f is not analytic at z_0.*

Example 13.3. Let $f(z) = \overline{z} = x - iy$ so that

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y.$$

We find

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1, & \frac{\partial v}{\partial y} &= -1, \\ \frac{\partial v}{\partial x} &= 0, & \frac{\partial u}{\partial y} &= 0. \end{aligned}$$

Since the Cauchy-Riemann equations are not satisfied for any z_0, we conclude that f is nowhere differentiable.

Example 13.4. Let $f(z) = |z|^2 = x^2 + y^2$ so that

$$u(x, y) = x^2 \quad \text{and} \quad v(x, y) = y^2.$$

We find

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= 2x_0, & \frac{\partial v}{\partial y}(x_0, y_0) &= 0, \\ \frac{\partial v}{\partial x}(x_0, y_0) &= 0, & \frac{\partial u}{\partial y}(x_0, y_0) &= 2y_0. \end{aligned}$$

The Cauchy-Riemann equations are only satisfied at $z_0 = (x_0, y_0) = (0, 0)$. Since the Cauchy-Riemann equations are NOT satisfied at $z_0 \neq 0$, we conclude that f is not differentiable at $z_0 \in \mathbb{C} \setminus \{0\}$. Hence, f is not analytic at 0. It is very important to stress that we CANNOT use the Cauchy-Riemann equations to determine whether or not $f'(0)$ exists. (Using the definition of derivative, we showed in Example 11.3 that $f'(0) = 0$.)

Exercise 13.5. Use the Cauchy-Riemann equations to show that $f(z) = \operatorname{Im} z$ is nowhere differentiable.

Exercise 13.6. Use the Cauchy-Riemann equations to show that $f(z) = \operatorname{Re} z$ is nowhere differentiable.

The key observation is that Theorem 13.1 gives us a necessary condition for differentiability, namely if f is differentiable at z_0, then f satisfies the Cauchy-Riemann equations at z_0. It does not, however, give us a sufficient condition for a function to be differentiable. That is, it is possible for a function $f = u + iv$ to satisfy the Cauchy-Riemann equations at z_0, yet not be differentiable at z_0.

Exercise 13.7. Consider the function

$$f(z) = f(x+iy) = \begin{cases} \dfrac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2+y^2}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$, but that f is not differentiable at $z = 0$. (Hint: Consider $\Delta z \to 0$ along (i) the real axis, and (ii) the line $y = x$.)

Theorem 13.8. *Let $f(z)$ be defined in some neighbourhood D of the point $z_0 = x_0 + iy_0$. If the Cauchy-Riemann equations are satisfied at z_0, namely*

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \textit{and} \quad \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0),$$

and if

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

all exist in D and are continuous at z_0, then f is differentiable at z_0.

Example 13.9. Suppose that $f = u + iv$ is analytic in a domain D. Show that u satisfies *Laplace's equation* in D (assuming that u_{xx}, u_{yy}, v_{xy}, v_{yx} exist in D and are sufficiently smooth so that $v_{xy} = v_{yx}$).

Solution. Since $f(z) = u(z) + iv(z) = u(x,y) + iv(x,y)$ is analytic in D, we know the Cauchy-Riemann equations are satisfied at any $z_0 = x_0 + iy_0 \in D$. This means that

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0).$$

Taking the second partials of u with respect to x and y implies that

$$\frac{\partial^2 u}{\partial x^2}(x_0,y_0) = \frac{\partial^2 v}{\partial x \partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial^2 v}{\partial y \partial x}(x_0,y_0) = -\frac{\partial^2 u}{\partial y^2}(x_0,y_0)$$

and so

$$\frac{\partial^2 u}{\partial x^2}(x_0,y_0) + \frac{\partial^2 u}{\partial y^2}(x_0,y_0) = \frac{\partial^2 v}{\partial x \partial y}(x_0,y_0) - \frac{\partial^2 v}{\partial y \partial x}(x_0,y_0) = 0.$$

Definition. An *entire* function is one that is analytic in the entire complex plane.

Example 13.10. Show that the function $f(z) = e^z = e^x e^{iy} = e^x[\cos y + i \sin y]$ is entire. Also show that $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

Solution. If $f(z) = e^z = e^x e^{iy} = e^x[\cos y + i \sin y]$, then

$$\frac{\partial u}{\partial x}(x_0, y_0) = e^{x_0} \cos y_0, \quad \frac{\partial v}{\partial x}(x_0, y_0) = e^{x_0} \sin y_0,$$

and

$$\frac{\partial v}{\partial y}(x_0, y_0) = e^{x_0} \cos y_0, \quad \frac{\partial u}{\partial y}(x_0, y_0) = -e^{x_0} \sin y_0.$$

Observe that

$$\frac{\partial u}{\partial x}(z_0), \quad \frac{\partial u}{\partial y}(z_0), \quad \frac{\partial v}{\partial x}(z_0), \quad \frac{\partial v}{\partial y}(z_0)$$

exist for all $z_0 \in \mathbb{C}$ and are clearly continuous at z_0. Since the Cauchy-Riemann equations are also satisfied for every $z_0 \in \mathbb{C}$, we conclude from Theorem 13.8 that $f(z) = e^z$ is differentiable at every $z_0 \in \mathbb{C}$. Hence, e^z is necessarily analytic at every $z_0 \in \mathbb{C}$ so that e^z is entire. Observe that if $z \in \mathbb{C}$, then $e^z \neq 0$. This follows from the fact that $e^x > 0$ for every $x \in \mathbb{R}$ and $\cos y + i \sin y \neq 0$ for every $y \in \mathbb{R}$ (i.e., $\cos y$ and $\sin y$ are never simultaneously equal to 0). To finish the proof that $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, suppose that $w \in \mathbb{C} \setminus \{0\}$ and observe that

$$e^{\log |w|}(\cos(\operatorname{Arg} w) + i \sin(\operatorname{Arg} w)) = w.$$

In other words, if $z = \log |w| + i \operatorname{Arg} w$, then

$$e^z = e^{\log |w| + i \operatorname{Arg} w} = |w| e^{i \operatorname{Arg}(w)} = w.$$

Since $\cos y$ and $\sin y$ are periodic with period 2π, we conclude that

$$e^z = e^{z + 2\pi i}.$$

That is, e^z is periodic with period $2\pi i$. Since $\operatorname{Arg}(w) \in (-\pi, \pi]$, we therefore take the *fundamental region* for e^z to be

$$\{-\pi < \operatorname{Im} z \leq \pi\}$$

as shown in Figure 13.1.

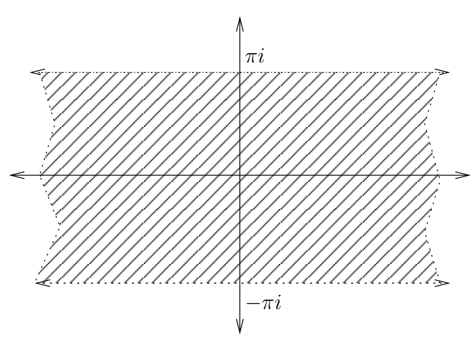

Figure 13.1: The fundamental region for e^z.

Mathematics 312 (Fall 2012) October 5, 2012
Prof. Michael Kozdron

Lecture #14: Harmonicity and the Cauchy-Riemann Equations

Suppose that $f(z) = u(z) + iv(z)$ is analytic in a domain D so that u and v satisfy the Cauchy-Riemann equations in D, namely

$$u_x(z_0) = v_y(z_0) \quad \text{and} \quad u_y(z_0) = -v_x(z_0)$$

for every $z_0 = x_0 + iy_0 \in D$. We know from Example 13.9 that if u_{xx}, u_{yy}, v_{xy}, v_{yx} exist in D and are sufficiently smooth so that $v_{xy} = v_{yx}$, then u satisfies *Laplace's equation* in D, namely

$$\frac{\partial^2 u}{\partial x^2}(x_0, y_0) + \frac{\partial^2 u}{\partial y^2}(x_0, y_0) = 0$$

for every $z_0 \in D$.

Definition. Suppose that $D \subseteq \mathbb{C}$ is a domain. We say that a function $u : D \to \mathbb{R}$ is *harmonic* if each of u_{xx}, u_{yy}, u_{xy}, and u_{yx} is continuous in D and if u satisfies Laplace's equation in D, namely

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = 0$$

for every $z_0 = x_0 + iy_0 \in D$.

Example 14.1. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is given by $u(z) = u(x, y) = x^3 - 3xy^2 + y$. Verify that u is harmonic in $\mathbb{C}$, and then find an analytic function $f : \mathbb{C} \to \mathbb{C}$ with $\operatorname{Re} f(z) = u(z)$.

Solution. To show that u is harmonic in $\mathbb{C}$, we need to show (i) u_{xx}, u_{yy}, u_{xy}, and u_{yx} are continuous, and (ii) $u_{xx} + u_{yy} = 0$. That is,

$$u_x = 3x^2 - 3y^2 \quad \text{so that} \quad u_{xx} = 6x \quad \text{and} \quad u_{yx} = -6y$$

and

$$u_y = -6xy + 1 \quad \text{so that} \quad u_{yy} = -6x \quad \text{and} \quad u_{xy} = -6y.$$

Clearly, u_{xx}, u_{yy}, u_{xy}, and u_{yx} are continuous and

$$u_{xx} + u_{yy} = 6x - 6x = 0$$

so that u is, in fact, harmonic in $\mathbb{C}$. To find an analytic function f with $\operatorname{Re} f(z) = u(z)$ means that we must find $v(z)$ such that $f(z) = u(z) + iv(z)$ is analytic in $\mathbb{C}$. Note that $v(z)$ is called a *harmonic conjugate* of $u(z)$. (As we will see shortly, $v(z)$ is not unique.) Since f is assumed to be analytic, we know that u and v must satisfy the Cauchy-Riemann equations. That is,

$$u_x = v_y \quad \text{implies} \quad v_y = 3x^2 - 3y^2$$

and

$$u_y = -v_x \quad \text{implies} \quad v_x = 6xy - 1.$$

Integrating v_y implies

$$v(x,y) = 3x^2y - 6y + C_1(x)$$

and integrating v_x implies that

$$v(x,y) = 3x^2y - x + C_2(y).$$

By comparing these two expressions for $v(x,y)$, we see that $v(x,y)$ must be of the form

$$v(x,y) = 3x^2y - 6y - x + C$$

where C is an arbitrary real constant. Since the problem asks us to find *one* analytic function f with $\operatorname{Re} f(z) = u(z)$, the one we'll choose is

$$f(z) = f(x,y) = u(x,y) + iv(x,y) = x^3 - 3xy^2 + y + i(3x^2y - 6y - x + 312).$$

It is worth noting that we can write $f(z)$ as a function of z as follows:

$$f(z) = z^3 - iz + 312i.$$

Remark. Assuming appropriate smoothness, we have shown that the real part of *every* analytic function f is harmonic. The converse, however, is not true. That is, not every smooth harmonic function $u : D \to \mathbb{R}$ is necessarily the real part of some analytic function. As an example, consider $u(z) = \log|z|$ for $z \in D = \{0 < |z| < 1\}$. It is not hard to show that u is harmonic in D. However, it can also be shown that u does *not* have a harmonic conjugate in D. Compare this to Problem #8 on Assignment #4. The function $u(z) = \log|z|$ for $z \in D = \{\operatorname{Re} z > 0\}$ is harmonic in D and does have a harmonic conjugate in D.

Analytic Properties of Elementary Functions

Recall from Lecture #13 that we set out to determine when a function is differentiable. One consequence of our calculations was the following. We showed that if f was differentiable at z_0, then f satisfied the Cauchy-Riemann equations at z_0. The way we derived this result was to compute $f'(z_0)$ in two ways and then equate real and imaginary parts. If we step back, however, we can view our computations as a way of calculating $f'(z_0)$.

Theorem 14.2. *Consider the function $f(z) = u(z) + iv(z)$ defined in some neighbourhood of z_0. If f is differentiable at $z_0 = x_0 + iy_0$, then*

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

and

$$f'(z_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Remark. It is important to stress that we must still know *a priori* that f is differentiable at z_0 in order to conclude that its derivative is given by either of these formulas. The most common way of doing this is to use Theorem 13.8.

Example 14.3. Prove that if $f(z) = e^z = e^x[\cos y + i \sin y]$, then $f'(z) = e^z$.

Solution. We know from Example 13.10 that e^z is entire. Therefore, we can apply Theorem 14.2 to conclude

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = e^{x_0} \cos y_0 + ie^{x_0} \sin y_0 = e^{z_0}.$$

Recall that we can write the real-valued functions $\sin\theta$ and $\cos\theta$ as

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

This motivates the following definition.

Definition. The complex-valued functions $\cos z$ and $\sin z$ are defined to be

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We now make a couple of important observations.

- The function e^z is periodic with period $2\pi i$ and the function e^{iz} is periodic with period 2π.
- Since e^{iz} and e^{-iz} are both entire functions, the functions $\cos z$ and $\sin z$ are also entire.
- $\sin(z + 2\pi k) = \sin z$ and $\cos(z + 2\pi k) = \cos z$ for any integer k. This means that the fundamental region for $\cos z$ and $\sin z$ is $\{0 \le \operatorname{Re} z < 2\pi\}$; see Figure 14.1.

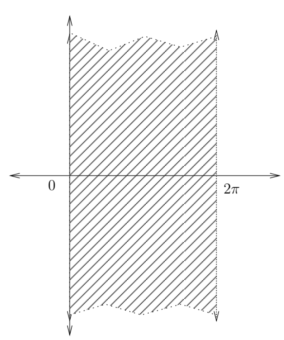

Figure 14.1: The fundamental region for $\cos z$ and $\sin z$.

Example 14.4. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \cos z \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z} \cos z = -\sin z.$$

Solution. We find

$$\frac{\mathrm{d}}{\mathrm{d}z} \sin z = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z} \cos z = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

Lecture #15: Analytic Properties of Elementary Functions

Recall that we have defined the complex-valued functions e^z, $\cos z$, and $\sin z$. The other complex-valued trigonometric functions are defined in the same way as their real counterparts. That is,

- $\tan z = \dfrac{\sin z}{\cos z}$,
- $\sec z = \dfrac{1}{\cos z}$,
- $\csc z = \dfrac{1}{\sin z}$, and
- $\cot z = \dfrac{1}{\tan z} = \dfrac{\cos z}{\sin z}$.

Note that $\cot z$ and $\csc z$ are analytic except at the zeroes of $\sin z$, namely at $z = k\pi$, $k \in \mathbb{Z}$. Also note that $\tan z$ and $\sec z$ are analytic except at the zeroes of $\cos z$, namely at $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$.

Exercise 15.1. Show that the following identities hold for $\cos z$ and $\sin z$:

- $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$,
- $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$,
- $\sin^2 z + \cos^2 z = 1$,
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$,
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_2 \sin z_1$,
- $\sin(2z) = 2 \sin z \cos z$, $\cos(2z) = \cos^2 z - \sin^2 z$.

In fact, we can also show that the differentiation formulas that hold for real-valued trigonometric functions also hold for the complex-valued ones; that is,

- $\dfrac{\mathrm{d}}{\mathrm{d}z} \tan z = \sec^2 z$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z} \cot z = -\csc^2 z$,
- $\dfrac{\mathrm{d}}{\mathrm{d}z} \sec z = \sec z \tan z$, and
- $\dfrac{\mathrm{d}}{\mathrm{d}z} \csc z = -\csc z \cot z$.

Exercise 15.2. Verify the previous differentiation formulas hold.

We are now about to define the complex-valued logarithm function. Recall that for real variables, we can define the *(natural) logarithm* of $x \neq 0$, written as $\log x$, to be that unique number satisfying $e^{\log x} = x$. Moreover, we also know that $\log(e^x) = x$ so that the functions $f(x) = e^x$ and $g(x) = \log x$ are inverses.

Example 15.3. Solve $e^x = \pi/4$ for $x \in \mathbb{R}$.

Solution. We can use logarithms to solve this problem. That is, $e^x = \pi/4$ implies $x = \log(e^x) = \log(\pi/4)$.

Remark. To solve the previous problem we used a key fact about real-valued logarithms, namely

$$e^{x_1} = e^{x_2} \quad \text{if and only if} \quad x_1 = x_2,$$

or, equivalently,

$$\log x_1 = \log x_2 \quad \text{if and only if} \quad x_1 = x_2.$$

We have already discovered that the function e^z is $2\pi i$ periodic, namely $e^z = e^{z+2\pi i}$, so that we cannot simply define the complex-valued logarithm to be the inverse of e^z.

Example 15.4. Solve $e^z = (1+i)/\sqrt{2}$ for $z \in \mathbb{C}$.

Solution. We write $(1+i)/\sqrt{2}$ in polar coordinates as $(1+i)/\sqrt{2} = e^{i\pi/4}$ so that we need to solve

$$e^z = e^{i\pi/4}$$

for z. Hence, one solution is $z = i\pi/4$. But this is not the only solution. By periodicity, we know $e^z = e^{z+2\pi ki}$, $k \in \mathbb{Z}$. Hence,

$$e^{z+2\pi ki} = e^{i\pi/4}$$

implies

$$z = (\pi/4 + 2\pi k)i, \quad k \in \mathbb{Z}.$$

Let $w \in \mathbb{C} \neq 0$. We know that there are infinitely many values of $z \in \mathbb{C}$ such that $e^z = w$; see Figure 15.1.

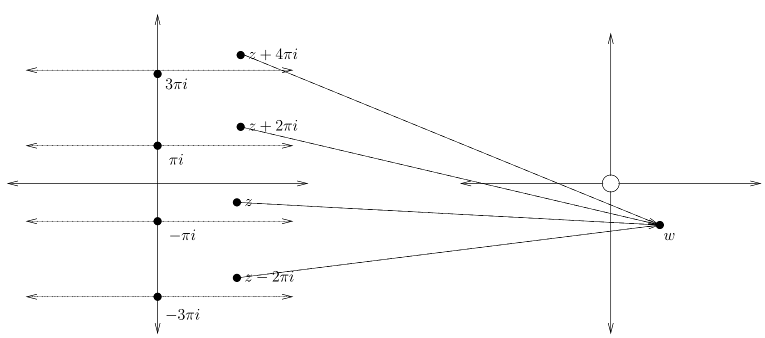

Figure 15.1: The image of $\mathbb{C}$ under the mapping e^z.

However, there is a *unique* value of z in the fundamental region $\{-\pi < \operatorname{Im} z \leq \pi\}$ with $e^z = w$. This is what we will use to define the logarithm of w; more precisely, this will be the *principal value of the logarithm.*

Definition. Suppose that $w \in \mathbb{C} \setminus \{0\}$. We define the *principal value of the logarithm of w*, denoted $\operatorname{Log} w$, to be

$$\operatorname{Log} w = \log |w| + i \operatorname{Arg}(w).$$

Remark. We are writing Log with a capital L to stress that it is the principal value of the complex-valued logarithm. Note that $\log x$ for $x \in \mathbb{R}$ denotes the usual real-valued natural logarithm.

Remark. The principal value of the logarithm of $w \neq 0$ can also be defined as the unique value of z with $-\pi < \operatorname{Im} z \leq \pi$ such that $e^z = w$.

Example 15.5. Compute $\operatorname{Log}(1+i)$.

Solution. Since $|1+i| = \sqrt{2}$ and $\operatorname{Arg}(1+i) = \pi/4$, we find

$$\operatorname{Log}(1+i) = \log \sqrt{2} + i\pi/4 = \frac{1}{2}\log 2 + i\frac{\pi}{4}.$$

Definition. Let $w \in \mathbb{C} \setminus \{0\}$. The *complex-valued logarithm of w* is the multiple-valued function given by

$$\log w = \log |w| + i \arg(w).$$

Since $\arg(w) = \{\operatorname{Arg}(w) + 2\pi k, k \in \mathbb{Z}\}$, we can also write

$$\log w = \{\log |w| + i \operatorname{Arg}(w) + 2\pi k i, k \in \mathbb{Z}\}.$$

Recall from Assignment #1 that $\arg(w_1 w_2) = \arg(w_1) + \arg(w_2)$ for all $w_1, w_2 \in \mathbb{C}$, but that $\operatorname{Arg}(w_1 w_2) \neq \operatorname{Arg}(w_1) + \operatorname{Arg}(w_2)$ for all $w_1, w_2 \in \mathbb{C}$. This translates into similar statements for the complex-valued logarithm and the principal value of the logarithm.

Exercise 15.6. Show that $\log(w_1 w_2) = \log w_1 + \log w_2$ for all $w_1, w_2 \in \mathbb{C} \setminus \{0\}$. Find values $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Log}(w_1 w_2) \neq \operatorname{Log} w_1 + \operatorname{Log} w_2$.

Proposition 15.7. *The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \operatorname{Log} z$ is continuous at all z except those along the negative real axis.*

Proof. Since $z \mapsto \log |z|$ is clearly continuous for all $z \in \mathbb{C} \setminus \{0\}$ and since $\operatorname{Log} z = \log |z| + i \operatorname{Arg}(z)$, the result follows from the fact (Assignment #3) that $z \mapsto \operatorname{Arg}(z)$ is discontinuous at each point on the nonpositive real axis. □

Recall that if $f : (0, \infty) \to \mathbb{R}$ is given by $f(x) = \log x$, then $f'(x) = 1/x$. The same type of formula holds for the principal value of the logarithm, but must be stated very carefully.

Theorem 15.8. *The function $z \mapsto \operatorname{Log} z$ is analytic in the domain $D = \mathbb{C} \setminus D^*$ where*

$$D^* = \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0 \ \textit{and} \ \operatorname{Im}(z) = 0\}$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z = \frac{1}{z}$$

for $z \in D$.

Proof. Let $w = \operatorname{Log} z$. We must show that

$$\lim_{z\to z_0} \frac{w - w_0}{z - z_0}$$

exists and equals $1/z_0$ for every $z_0 \in D$. However, we know (by definition of $\operatorname{Log} z$) that $z = e^w$. We also know from from Example 13.10 and Example 14.3 that $f(w) = e^w$ is entire with $f'(w) = e^w$. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}w} f(w)\bigg|_{w=w_0} = \frac{\mathrm{d}}{\mathrm{d}w} e^w\bigg|_{w=w_0} = \frac{\mathrm{d}z}{\mathrm{d}w}\bigg|_{w=w_0} = \lim_{w\to w_0} \frac{z - z_0}{w - w_0} = e^{w_0} = z_0. \tag{$*$}$$

The next step is to observe that by continuity (Proposition 15.7), $w \to w_0$ as $z \to z_0$. Hence,

$$\lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \lim_{w\to w_0} \frac{w - w_0}{z - z_0}. \tag{$**$}$$

However, compare the right side of $(**)$ with $(*)$ to conclude

$$\frac{\mathrm{d}}{\mathrm{d}z} \operatorname{Log} z\bigg|_{z=z_0} = \lim_{z\to z_0} \frac{w - w_0}{z - z_0} = \lim_{w\to w_0} \frac{w - w_0}{z - z_0} = \lim_{w\to w_0} \frac{1}{\frac{z-z_0}{w-w_0}} = \frac{1}{z_0}$$

for every $z_0 \in D$. $\square$

Lecture #16: Applications of the Cauchy-Riemann Equations

Example 16.1. Prove that if r and θ are polar coordinates, then the functions $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ (where n is a positive integer) are harmonic as functions of x and y.

Solution. Consider $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ where n is a positive integer. The key observation is that de Moivre's formula tells us these are the real and imaginary parts, respectively, of $(r\cos\theta + ir\sin\theta)^n$; that is, if $z = x + iy = re^{i\theta}$, then

$$z^n = r^n e^{in\theta} = r^n \cos(n\theta) + ir^n \sin(n\theta).$$

Hence, let $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$. In order to show that u and v are harmonic as functions of x and y, we can use Example 13.9 which tells us that the real part of an analytic function is harmonic (assuming the partial derivatives are smooth enough). However, Example 13.9 says nothing about the imaginary part of an analytic function. Thus, the first step is to prove the following.

Proposition 16.2. *If $f = u + iv$ is analytic in a domain D, then v satisfies Laplace's equation in D (assuming that v_{xx}, v_{yy}, u_{xy}, u_{yx} exist in D and are sufficiently smooth so that $u_{xy} = u_{yx}$).*

Proof. Since $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ is analytic in D, we know the Cauchy-Riemann equations are satisfied at any $z_0 = x_0 + iy_0 \in D$. This means that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0).$$

Taking the second partials of v with respect to x and y implies that

$$\frac{\partial^2 v}{\partial y^2}(x_0, y_0) = \frac{\partial^2 u}{\partial y \partial x}(x_0, y_0) \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}(x_0, y_0) = -\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0)$$

and so

$$\frac{\partial^2 v}{\partial x^2}(x_0, y_0) + \frac{\partial^2 v}{\partial y^2}(x_0, y_0) = -\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0) + \frac{\partial^2 u}{\partial y \partial x}(x_0, y_0) = 0$$

which completes the proof. □

Therefore, we see that <u>**if**</u> we can show that $f(z) = z^n$ is analytic, we can conclude for free from Example 13.9 and this proposition that $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$ are harmonic as functions of x and y.

In order to prove that $f(z) = z^n$ is analytic, however, we need to show that $f'(z_0)$ exists for all $z_0 \in \mathbb{C}$. Consider

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z}.$$

By the binomial theorem,

$$(z_0 + \Delta z)^n = \sum_{j=0}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^j = z_0^n + n z_0^{n-1} \Delta z + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^j,$$

and so

$$\frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z} = n z_0^{n-1} + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^{j-1}.$$

Since $j - 1 \geq 0$ for $2 \leq j \leq n$, we immediately deduce that

$$\lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z} = \lim_{\Delta z \to 0} \left[n z_0^{n-1} + \sum_{j=2}^{n} \binom{n}{j} z_0^{n-j} (\Delta z)^{j-1} \right] = n z_0^{n-1}$$

proving $f(z) = z^n$ is entire with $f'(z_0) = n z_0^{n-1}$ for all $z_0 \in \mathbb{C}$. In particular, $u = \mathrm{Re}(z^n) = r^n \cos(n\theta)$ and $v = \mathrm{Im}(z^n) = r^n \sin(n\theta)$ are both harmonic as functions of x and y.

The Cauchy-Riemann Equations and Laplace's Equation in Polar Coordinates

An equivalent way to solve Example 16.1 is to compute $u_{xx} + u_{yy}$ and $v_{xx} + v_{yy}$ directly for both $u = r^n \cos(n\theta)$ and $v = r^n \sin(n\theta)$. The difficulty with this approach is that u and v, as written, are functions of r and θ, but the partials that we wish to compute are with respect to x and y. Therefore, we must use the multivariable chain rule to determine u_r, u_θ, v_r, v_θ in terms of u_x, u_y, v_x, v_y. That is, we will introduce a change of variables

$$U(r, \theta) = u(x, y) \quad \text{and} \quad V(r, \theta) = v(x, y)$$

with $x = r\cos\theta$ and $y = r\sin\theta$. Observe that $r^2 = x^2 + y^2$ so that $2rr_x = 2x$ which implies

$$r_x = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

Moreover, $\tan\theta = y/x$ so that $\theta_x = \sec^2\theta = -y/x^2$ which implies

$$\theta_x = -\frac{y}{x^2 \sec^2\theta} = -\frac{y\cos^2\theta}{x^2} = -\frac{r\sin\theta\cos^2\theta}{r^2\sin^2\theta} = -\frac{\sin\theta}{r}.$$

Similarly,

$$r_y = \sin\theta \quad \text{and} \quad \theta_y = \frac{\cos\theta}{r}.$$

By the chain rule, we now find

$$u_x = U_r r_x + U_\theta \theta_x = (\cos\theta) U_r + (-r^{-1}\sin\theta) U_\theta, \quad u_y = U_r r_y + U_\theta \theta_y = (\sin\theta) U_r + (r^{-1}\cos\theta) U_\theta,$$

and

$$v_x = V_r r_x + V_\theta \theta_x = (\cos\theta) V_r + (-r^{-1}\sin\theta) V_\theta, \quad v_y = V_r r_y + V_\theta \theta_y = (\sin\theta) V_r + (r^{-1}\cos\theta) V_\theta.$$

If we now assume that $f(z) = u(z) + iv(z) = U(r,\theta) + iV(r,\theta)$ is differentiable at $z_0 = r_0 e^{i\theta_0}$ so that the Cauchy-Riemann equations are satisfied at z_0, then

$$u_x(z_0) = v_y(z_0) \quad \text{and} \quad u_y(z_0) = -v_x(z_0).$$

This implies

$$(\cos\theta_0)U_r(r_0,\theta_0) - (r_0^{-1}\sin\theta_0)U_\theta(r_0,\theta_0) = (\sin\theta_0)V_r(r_0,\theta_0) + (r_0^{-1}\cos\theta_0)V_\theta(r_0,\theta_0) \qquad (*)$$

and

$$(\sin\theta_0)U_r(r_0,\theta_0) + (r_0^{-1}\cos\theta_0)U_\theta(r_0,\theta_0) = -(\cos\theta_0)V_r(r_0,\theta_0) + (r_0^{-1}\sin\theta_0)V_\theta(r_0,\theta_0). \qquad (**)$$

Simplifying $(*)$ and $(**)$ yields

$$(U_r(r_0,\theta_0) - r_0^{-1}V_\theta(r_0,\theta_0))\cos\theta_0 - (V_r(r_0,\theta_0) + r_0^{-1}U_\theta(r_0,\theta_0))\sin\theta_0 = 0 \qquad (\dagger)$$

and

$$(V_r(r_0,\theta_0) + r_0^{-1}U_\theta(r_0,\theta_0))\cos\theta_0 + (U_r(r_0,\theta_0) - r_0^{-1}V_\theta(r_0,\theta_0))\sin\theta_0 = 0. \qquad (\ddagger)$$

If we then multiple $(\dagger)$ by $\cos\theta_0$ and $(\ddagger)$ by $\sin\theta_0$, and then add, we obtain

$$(U_r(r_0,\theta_0) - r_0^{-1}V_\theta(r_0,\theta_0))(\cos^2\theta_0 + \sin^2\theta_0) = 0$$

which implies $U_r(r_0,\theta_0) = r_0^{-1}V_\theta(r_0,\theta_0)$. On the other hand, if we then multiple $(\dagger)$ by $-\sin\theta_0$ and $(\ddagger)$ by $\cos\theta_0$, and then add, we obtain

$$(V_r(r_0,\theta_0) + r_0^{-1}U_\theta(r_0,\theta_0))(\cos^2\theta_0 + \sin^2\theta_0) = 0$$

which implies $r_0^{-1}U_\theta(r_0,\theta_0) = -V_r(r_0,\theta_0)$.

Theorem 16.3. *Let $z = re^{i\theta}$. If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is differentiable at $z_0 = r_0 e^{i\theta_0}$, then the Cauchy-Riemann equations in polar coordinates are satisfied at z_0; that is,*

$$\frac{\partial U}{\partial r}(r_0,\theta_0) = \frac{1}{r_0}\frac{\partial V}{\partial \theta}(r_0,\theta_0) \quad \textit{and} \quad \frac{1}{r_0}\frac{\partial U}{\partial \theta}(r_0,\theta_0) = -\frac{\partial V}{\partial r}(r_0,\theta_0).$$

Summary. The Cauchy-Riemann equations in polar coordinates can be remembered as

$$\boxed{U_r = \frac{1}{r}V_\theta \quad \text{and} \quad \frac{1}{r}U_\theta = -V_r.}$$

Example 16.4. Suppose that $U(r,\theta) = r^n\cos(n\theta)$ and $V(r,\theta) = r^n\sin(n\theta)$. We find

$$U_r = nr^{n-1}\cos(n\theta)$$
$$V_\theta = nr^n\cos(n\theta)$$

and

$$U_\theta = -nr^n\sin(n\theta)$$
$$V_r = nr^{n-1}\sin(n\theta)$$

so that $U_r = r^{-1}V_\theta$ and $r^{-1}U_\theta = -V_r$. Hence, U and V satisfy the Cauchy-Riemann equations in polar coordinates.

We can now use the Cauchy-Riemann equations to derive Laplace's equation in polar coordinates. (Assume that all second partials exist and are sufficiently smooth so that the mixed partials are equal.) That is, we know

$$u_x = v_y \quad \text{implies} \quad rU_r = V_\theta \quad \text{and} \quad u_y = -v_x \quad \text{implies} \quad U_\theta = -rV_r$$

and so taking derivatives with respect to x of the first equation and derivatives with respect to y of the second equation implies

$$0 = (u_x - v_y)_x + (u_y + v_x)_y = (rU_r - V_\theta)_x + (U_\theta + rV_r)_y.$$

Now, using the chain rule, we find

$$(rU_r - V_\theta)_x = r_x U_r + r(U_{rr} r_x + U_{\theta r}\theta_x) - (V_{\theta\theta}\theta_x + V_{r\theta} r_x)$$

and

$$(U_\theta + rV_r)_y = (U_{\theta\theta}\theta_y + U_{r\theta} r_y) + r_y V_r + r(V_{rr} r_y + V_{\theta r}\theta_y).$$

Adding the previous two terms, using the equality of the mixed partials, and simplifying implies

$$r_x U_r + r r_x U_{rr} + (r\theta_x + r_y)U_{\theta r} + \theta_y U_{\theta\theta} = -r_y V_r - r r_y V_{rr} - (r\theta_y - r_x)V_{r\theta} + \theta_x V_{\theta\theta}. \qquad (*)$$

The next step is to note that

$$r\theta_x + r_y = r \cdot -\frac{\sin\theta}{r} + \sin\theta = 0 \quad \text{and} \quad r\theta_y - r_x = r \cdot \frac{\cos\theta}{r} - \cos\theta = 0.$$

so that $(*)$ becomes

$$r_x U_r + r r_x U_{rr} + \theta_y U_{\theta\theta} = -r_y V_r - r r_y V_{rr} + \theta_x V_{\theta\theta}.$$

Substituting in r_x, θ_x, r_y, θ_y, we conclude

$$\cos\theta \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right] = -\sin\theta \left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta} \right]. \qquad (\dagger)$$

If, instead, at the beginning of the derivation we had taken derivatives with respect to y of the first equation and derivatives with respect to x of the second equation, then we would have found

$$\cos\theta \left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta} \right] = -\sin\theta \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right]. \qquad (\ddagger)$$

We now multiple $(\dagger)$ by $\cos\theta$, multiply $(\ddagger)$ by $\sin\theta$, and add, then we conclude

$$(\cos^2\theta + \sin^2\theta) \left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta} \right] = 0$$

and so we finally arrive at Laplace's equation in polar coordinates

$$\boxed{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0.}$$

Note that we can also conclude immediately that V satisfies Laplace's equation in polar coordinates as well,

$$V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta} = 0.$$

Example 16.5. Suppose that $U(r, \theta) = r^n \cos(n\theta)$. We can now show directly that U is harmonic. That is,

$$U_r = nr^{n-1}\cos(n\theta), \quad U_{rr} = n(n-1)r^{n-2}\cos(n\theta), \quad U_\theta = -nr^n \sin(n\theta), \quad U_{\theta\theta} = -n^2 r^n \cos(n\theta)$$

so that

$$\begin{aligned}
U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} &= n(n-1)r^{n-2}\cos(n\theta) + \frac{1}{r} \cdot nr^{n-1}\cos(n\theta) + \frac{1}{r^2} \cdot -n^2 r^n \cos(n\theta) \\
&= r^{n-2}\cos(n\theta)[n(n-1) + n - n^2] \\
&= 0
\end{aligned}$$

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Prof. Michael Kozdron

Lecture #17: Contour Integration

A *contour integral* is just a two-dimensional line integral (also known as a path integral).

A *curve* in the complex plane will be denoted by $z = z(t)$, $a \le t \le b$. In other words, the function $z : \mathbb{R} \to \mathbb{C}$ given by $t \mapsto z(t)$, $a \le t \le b$, describes a curve in $\mathbb{C}$.

A *smooth curve* $z = z(t)$ is a curve such that

(i) $z(t)$ has a continuous derivative,

(ii) $z'(t) \neq 0$ for all $t \in [a, b]$,

(iii) $z(t)$ is a one-to-one function.

Example 17.1. In Figure 17.1 below are examples of a smooth curve (left) and a non-smooth curve (right).

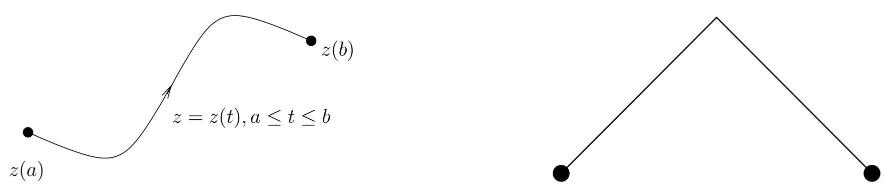

Figure 17.1: Figure for Example 17.1. The curve on the left is smooth, whereas the curve on the right is not smooth.

Example 17.2. Parametrize C_1, the line segment going from 1 to $2 + i$ in $\mathbb{C}$ as shown in Figure 17.2 below.

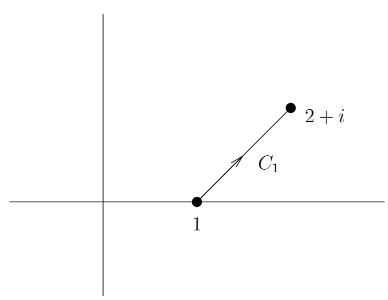

Figure 17.2: Figure for Example 17.2 and Example 17.4.

Solution. Let $x(t) = 1 + t$, $0 \le t \le 1$, and let $y(t) = t$, $0 \le t \le 1$, so that

$$z(t) = x(t) + iy(t) = 1 + t + it = 1 + (1 + i)t$$

for $0 \le t \le 1$.

Definition. A *contour* is a finite sequence of concatenated smooth curves $z = z(t)$ with a specified direction.

Example 17.3. Figure 17.3 below shows an example of a contour. Note that this particular contour is the concatenation of two smooth curves.

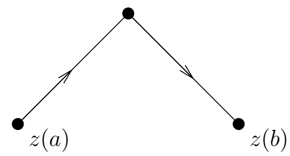

Figure 17.3: Figure for Example 17.3; an example of a contour.

Let C be a contour and consider

$$I = \int_C f(z)\,\mathrm{d}z$$

which is the contour integral of the function $f(z)$ along the contour C. That is, we integrate $f(z)$ along C in $\mathbb{C}$. Let $z = z(t)$, $a \le t \le b$, be a smooth parametrization of C. We define

$$I = \int_C f(z)\,\mathrm{d}z$$

to equal

$$I = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t.$$

Note that this definition requires

$$z'(t) = \frac{\mathrm{d}z(t)}{\mathrm{d}t}$$

to exist.

Example 17.4. Compute

$$I_1 = \int_{C_1} z^2\,\mathrm{d}z$$

where C_1 is the line segment going from 1 to $2 + i$ in $\mathbb{C}$ as shown in Figure 17.2 above.

Solution. We know from Example 17.1 that C_1 is parametrized by $z(t) = 1 + (1+i)t$, $0 \le t \le 1$. Note that $z(0) = 1$ and $z(1) = 2 + i$. Now

$$z(t)^2 = [1 + (1+i)t]^2 = 1 + 2(1+i)t + (1+i)^2t^2 \quad \text{and} \quad z'(t) = 1 + i$$

so that

$$\begin{aligned}
\int_{C_1} z^2\,\mathrm{d}z = \int_0^1 z(t)^2 \cdot z'(t)\,\mathrm{d}t &= (1+i)\int_0^1 1 + 2(1+i)t + (1+i)^2t^2\,\mathrm{d}t \\
&= \left[(1+i)t + (1+i)^2t^2 + \frac{(1+i)^3}{3}t^3\right]_0^1 \\
&= (1+i) + (1+i)^2 + \frac{(1+i)^3}{3} \\
&= \frac{1}{3} + \frac{11}{3}i.
\end{aligned}$$

Example 17.5. Compute

$$I_{23} = \int_{C_{23}} z^2 \, \mathrm{d}z$$

where C_2 is the line segment going from 1 to 2 along the real axis, C_3 is the line segment going from 2 to $2+i$ parallel to the imaginary axis, and $C_{23} = C_2 \oplus C_3$ as shown in Figure 17.4 below.

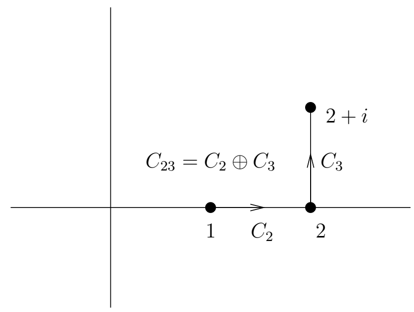

Figure 17.4: Figure for Example 17.5.

Solution. We can parametrize C_2 as follows. Let $x(t) = 1 + t$, $0 \le t \le 1$, and let $y(t) = 0$, $0 \le t \le 1$, so that

$$z(t) = x(t) + iy(t) = 1 + t$$

for $0 \le t \le 1$. We can parametrize C_3 as follows. Let $x(t) = 2$, $0 \le t \le 1$, and let $y(t) = t$, $0 \le t \le 1$, so that $z(t) = 2 + it$, $0 \le t \le 1$. Note, though, that we want to concatenate C_2 and C_3. Therefore, we will reparametrize C_3 by $z(t) = 2 + i(t-1) = 2 - i + it$, $1 \le t \le 2$, so that $C_{23} = C_2 \oplus C_3$ is parametrized by

$$z(t) = \begin{cases} 1 + t, & 0 \le t \le 1, \\ 2 - i + it, & 1 \le t \le 2. \end{cases}$$

Now,

$$\begin{aligned} \int_{C_{23}} z^2 \, \mathrm{d}z = \int_0^2 z(t)^2 \cdot z'(t) \, \mathrm{d}t &= \int_0^1 z(t)^2 \cdot z'(t) \, \mathrm{d}t + \int_1^2 z(t)^2 \cdot z'(t) \, \mathrm{d}t \\ &= \int_0^1 (1+t)^2 \cdot 1 \, \mathrm{d}t + \int_1^2 (2 - i + it)^2 \cdot i \, \mathrm{d}t \\ &= \frac{(1+t)^3}{3} \bigg|_0^1 + \frac{(2 - i + it)^3}{3} \bigg|_1^2 \\ &= \frac{8}{3} - \frac{1}{3} + \frac{(2+i)^3}{3} - \frac{8}{3} \\ &= \frac{(2+i)^3}{3} - \frac{1}{3} \\ &= \frac{1}{3} + \frac{11}{3} i. \end{aligned}$$

Observe that our answers from Examples 17.4 and 17.5 are the same; that is,

$$I_1 = I_{23} = \frac{1}{3} + \frac{11}{3}i.$$

Is this a coincidence? In other words, we have taken two distinct contours connecting the same beginning and ending points, and found that the answer to both contour integrals is the same. Suppose we take more complicated contours connecting the same same beginning and ending points. Will we get the same value for any contour integral?

Example 17.6. Compute

$$I_1 = \int_{C_1} \overline{z}\,\mathrm{d}z$$

if $C_1 = \{e^{it}, 0 \le t \le \pi\}$ is that part of the upper half of the unit circle going from 1 to -1.

Solution. If $z(t) = e^{it}$, $0 \le t \le \pi$, then $z'(t) = ie^{it}$, and so

$$\int_{C_1} \overline{z}\,\mathrm{d}z = \int_0^{\pi} \overline{z(t)} \cdot z'(t)\,\mathrm{d}t = \int_0^{\pi} e^{-it} \cdot ie^{it}\,\mathrm{d}t = i\int_0^{\pi} \mathrm{d}t = i\pi.$$

Example 17.7. Compute

$$I_2 = \int_{C_2} \overline{z}\,\mathrm{d}z$$

where $C_2 = \{e^{-it}, 0 \le t \le \pi\}$ is that part of the lower half of the unit circle going from 1 to -1.

Solution. If $z(t) = e^{-it}$, $0 \le t \le \pi$, then $z'(t) = -ie^{it}$, and so

$$\int_{C_2} \overline{z}\,\mathrm{d}z = \int_0^{\pi} \overline{z(t)} \cdot z'(t)\,\mathrm{d}t = \int_0^{\pi} e^{it} \cdot -ie^{-it}\,\mathrm{d}t = -i\int_0^{\pi} \mathrm{d}t = -i\pi.$$

Note that the answers to the previous two examples are different; that is, even though the contours C_1 and C_2 start and end at the same points, $I_1 \neq I_2$. What is the difference between this pair of examples and the previous pair of examples?

Theorem 17.8 (Fundamental Theorem of Calculus for Contour Integrals)**.** *Suppose that D is a domain. If $f(z)$ is continuous in D and has an antiderivative $F(z)$ throughout D (i.e., $F(z)$ is analytic in D with $F'(z) = f(z)$ for every $z \in D$), then*

$$\int_C f(z)\,\mathrm{d}z = F(z(b)) - F(z(a))$$

for any contour C lying entirely in D.

Proof. Suppose that C lies entirely in D and is parametrized by $z = z(t)$, $a \le t \le b$. From the definition of contour integral, we have

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t$$

and note that the assumption that $f(z)$ is continuous means that $f(z(t)) \cdot z'(t)$ is Riemann integrable on $[a, b]$. The assumption that f has an antiderivative F means that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t).$$

Therefore,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(z(t))\,\mathrm{d}t = F(z(b)) - F(z(a))$$

by the usual Fundamental Theorem of Calculus. □

Example 17.9. Compute

$$\int_C z^2\,\mathrm{d}z$$

where C is any contour connecting 1 and $2 + i$.

Solution. Observe that $f(z) = z^2$ is continuous in $\mathbb{C}$ and $F(z) = z^3/3$ is entire with $F'(z) = f(z)$. Therefore, if C is any contour with $z(a) = 1$ and $z(b) = 2 + i$, then the Fundamental Theorem of Calculus for Contour Integrals implies

$$\int_C z^2\,\mathrm{d}z = \left.\frac{z^3}{3}\right|_{z=2+i} - \left.\frac{z^3}{3}\right|_{z=1} = \frac{(2+i)^3}{3} - \frac{1}{3} = \frac{1}{3} + \frac{11}{3}i.$$

Remark. This explains why the answers to Examples 17.4 and 17.5 are the same. Note that the function from Examples 17.6 and 17.7, namely $\bar{z}$, does not have an antiderivative. This is why the Fundamental Theorem of Calculus for Contour Integrals does not apply, and so we are not surprised that contour integrals of $\bar{z}$ *do* depend on the contour taken.

Example 17.10. Compute

$$\int_C e^{iz}\,\mathrm{d}z$$

where C is that part of the unit circle in the first quadrant going from 1 to i.

Solution. Observe that $f(z) = e^{iz}$ is continuous in $\mathbb{C}$ and $F(z) = -ie^{iz}$ is entire with $F'(z) = f(z)$. Therefore, since C is a contour with $z(a) = 1$ and $z(b) = i$, the Fundamental Theorem of Calculus for Contour Integrals implies

$$\int_C e^{iz}\,\mathrm{d}z = -ie^{iz}\Big|_{z=i} + ie^{iz}\Big|_{z=1} = -ie^{-1} + ie^{i} = ie^{i} - ie^{-1}.$$

Lecture #18: The Cauchy Integral Theorem

The goal for today's class will be to investigate the conditions under which

$$\int_C f(z)\,\mathrm{d}z = 0$$

for a closed contour C.

Theorem 18.1 (Fundamental Theorem of Calculus for Integrals over Closed Contours)**.** *Suppose that D is a domain. If $f(z)$ is continuous in D and has an antiderivative $F(z)$ throughout D (i.e., $F(z)$ is analytic in D with $F'(z) = f(z)$ for every $z \in D$), then*

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D.

Proof. This follows from the usual Fundamental Theorem of Calculus. Suppose that C is parametrized by $z = z(t)$, $a \le t \le b$. The hypothesis that $f(z)$ is continuous in D is necessary for the contour integral

$$\int_C f(z)\,\mathrm{d}z$$

to equal the Riemann integral

$$\int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t.$$

The assumption that f has an antiderivative F means that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(z(t)) = F'(z(t)) \cdot z'(t) = f(z(t)) \cdot z'(t).$$

Therefore,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t)) \cdot z'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(z(t))\,\mathrm{d}t = F(z(b)) - F(z(a))$$

by the usual Fundamental Theorem of Calculus. The assumption that C is a closed contour means that $z(a) = z(b)$ which implies $F(z(b)) = F(z(a))$. Hence,

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D. □

Remark. This theorem can apply if D is an annulus and C surrounds the hole.

Theorem 18.2 (Cauchy Integral Theorem, Basic Version)**.** *Suppose that D is a domain. If $f(z)$ is analytic in D and $f'(z)$ is continuous throughout D, then*

$$\int_C f(z)\,\mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that the region surrounded by C is a simply connected subdomain of D (in other words, C is continuously deformable to a point.)

This follows from Green's theorem and requires the assumptions that $f'(z)$ be continuous throughout D and C be continuously deformable to a point. Recall that Green's theorem is usually stated as follows.

Theorem 18.3 (Green's Theorem)**.** *Suppose that R is a simply connected domain and that $C = \partial R$ is a closed contour oriented counterclockwise. Let $P = P(x,y) : R \to \mathbb{R}$, $Q = Q(x,y) : R \to \mathbb{R}$ be continuously differentiable in R (so that P_x, P_y, Q_x, Q_y are continuous in R). Then,*

$$\int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.$$

Proof of Cauchy Integral Theorem, Basic Version. Suppose that D is a domain and C is a closed contour in D which is continuously deformable to a point so that R, the interior of C, is a simply connected subdomain of D. Suppose further that $f(z)$ is analytic in D and that $f'(z)$ is continuous in D. In particular, this means that $f(z)$ is analytic in R, and that $f'(z)$ is continuous in R. Thus, if we write $f(z) = u(x,y) + iv(x,y)$ for $z \in D$ and $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, then

$$\begin{aligned}
\int_C f(z)\,\mathrm{d}z &= \int_C (u(x,y) + iv(x,y))(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \int_C u(x,y)\,\mathrm{d}x + iv(x,y)\,\mathrm{d}x + iu(x,y)\,\mathrm{d}y - v(x,y)\,\mathrm{d}y \\
&= \int_C u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y + i\int_C v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y.
\end{aligned}$$

Without loss of generality, assume that C is oriented counterclockwise. Since $C = \partial R$ is a closed contour, R is a simply connected domain, and u_x, u_y, v_x, v_y are continuous (since $f(z)$ is analytic in R and $f'(z)$ is continuous in R), we can apply Green's theorem to each integral separately. That is,

$$\begin{aligned}
\int_C u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y &= \iint_R \left(-\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y \\
&= -\iint_R \left(\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y \qquad (*)
\end{aligned}$$

and

$$\int_C v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y. \qquad (**)$$

However, since $f(z)$ is analytic in D we know that the Cauchy-Riemann equations are satisfied in D; that is,

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

for any $z_0 = x_0 + iy_0 \in D$. This implies that

$$\iint_R \left(\frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = 0 \quad \text{and} \quad \iint_R \left(\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = 0$$

so that $(*)$ and $(**)$ imply

$$\int_C f(z) \, \mathrm{d}z = \int_C u(x,y) \, \mathrm{d}x - v(x,y) \, \mathrm{d}y + i \int_C v(x,y) \, \mathrm{d}x + u(x,y) \, \mathrm{d}y = 0$$

as required. $\square$

Remark. This theorem cannot apply if D is an annulus and C surrounds the hole.

Example 18.4. Suppose that $D = \{1 < |z| < 3\}$ is the interior of the annulus of inner radius 1 and outer radius 3. Let $C = \{|z| = 2\}$ denote the circle of radius 2. Show

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Solution. Observe that D is a domain. Also observe that $f(z) = 3z^2$ is continuous in D and has antiderivative $F(z) = z^3$ throughout D; that is, $F(z) = z^3$ is analytic in D with $F'(z) = 3z^2 = f(z)$. The contour C is closed and lies entirely in D. Thus, the hypotheses have been met for the Fundamental Theorem of Calculus for Integrals over Closed Contours and so we conclude

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Note that we **cannot** apply the Cauchy Integral Theorem to solve this problem. It is true that $f(z) = 3z^2$ is analytic in D with $f'(z) = 6z$ so that $f'(z)$ is continuous in D. It is also true that the contour C lies entirely in D. However, C is *not* continuously deformable to a point; in other words, the interior of C is *not* a simply connected subdomain of D. In fact, if $R \subset D$ denotes the interior of C, then $R = \{1 < |z| < 2\}$. Thus, the hypotheses for the Cauchy Integral Theorem have not been met.

Example 18.5. Suppose that $D = \{|z| < 3\}$ is the interior of the disk of radius 3. Let $C = \{|z| = 2\}$ denote the circle of radius 2. Show

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Solution. In this case, since D is a domain and the closed contour C is continuously deformable to a point, we can apply the Cauchy Integral Theorem. That is, $f(z) = 3z^2$ is analytic in D and the interior of C is $\{|z| < 2\}$ which is a simply connected subdomain of D. Therefore, by the Cauchy Integral Theorem, Basic Version, we conclude

$$\int_C 3z^2 \, \mathrm{d}z = 0.$$

Of course, we could also use the Fundamental Theorem of Calculus for Integrals over Closed Contours to draw the same conclusion. That is, $f(z) = 3z^2$ is continuous in D and has antiderivative $F(z) = z^3$ throughout D, so that

$$\int_C 3z^2 \, \mathrm{d}z = 0$$

since the hypotheses of Theorem 18.1 have been met.

Remark. The Cauchy Integral Theorem in the form we stated it was first proved by Augustin-Louis Cauchy (1789–1857). It was later shown by Édouard Goursat (1858–1936) that the assumption that $f'(z)$ be continuous is unnecessary.

Theorem 18.6 (Cauchy Integral Theorem, Advanced Version)**.** *Suppose that D is a domain. If $f(z)$ is analytic in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0$$

for any closed contour C lying entirely in D having the property that the region surrounded by C is a simply connected subset of D (in other words, C is continuously deformable to a point.)

Remark. The proof of this theorem is much too sophisticated for Math 312. However, for the purposes of this class, any time you are asked to use the Cauchy Integral Theorem, you will be able to verify that $f'(z)$ is continuous.

Corollary 18.7 (Cauchy Integral Theorem for Simply Connected Domains, Basic Version)**.** *Suppose that D is a simply connected domain. If $f(z)$ is analytic in D, $f'(z)$ is continuous in D, and C is a closed contour lying entirely in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0.$$

Corollary 18.8 (Cauchy Integral Theorem for Simply Connected Domains, Advanced Version)**.** *Suppose that D is a simply connected domain. If $f(z)$ is analytic in D and C is a closed contour lying entirely in D, then*

$$\int_C f(z) \, \mathrm{d}z = 0.$$

Proof of both corollaries. If D is simply connected, then any closed contour lying entirely in D is necessarily continuously deformable to a point. □

Mathematics 312 (Fall 2012) October 19, 2012
Prof. Michael Kozdron

Lecture #19: Applications of the Cauchy Integral Theorem

Example 19.1. Suppose $p \in \mathbb{Z}$. Compute

$$\int_C z^p \, \mathrm{d}z$$

where $C = \{|z| = r\}$ is the circle of radius $r > 0$ centred at 0 oriented counterclockwise.

Solution. We will consider separately two cases, namely (i) $p = 0, 1, 2, \ldots$, and (ii) $p = -1, -2, \ldots$.

In the first case, we can use either the Fundamental Theorem of Calculus for Integrals over Closed Contours or the Cauchy Integral Theorem to conclude

$$\int_C z^p \, \mathrm{d}z = 0.$$

To use the FTC, observe that $f(z) = z^p$, $p = 0, 1, 2, \ldots$, is continuous in $\mathbb{C}$ and that $F(z) = (p+1)^{-1} z^{p+1}$ is analytic in $\mathbb{C}$ with $F'(z) = z^p = f(z)$. Since C is a closed contour, the hypotheses of the FTC have been met so that

$$\int_C z^p \, \mathrm{d}z = 0.$$

Alternatively, the function $f(z) = z^p$, $p = 0, 1, 2, \ldots$, is analytic in $\mathbb{C}$ with $f'(z) = 0$ for $p = 0$ and $f'(z) = pz^{p-1}$ for $p = 1, 2, \ldots$. In any case, $f'(z)$ is continuous in $\mathbb{C}$. Thus, the hypotheses of the Cauchy Integral Theorem, Basic Version have been met and so we conclude

$$\int_C z^p \, \mathrm{d}z = 0$$

as before.

In the second case, consider the function $f(z) = z^p$, $p = -1, -2, \ldots$. This function is not defined at $z = 0$ and so it is necessarily not continuous at $z = 0$ and not analytic at $z = 0$. Thus, in order to compute

$$\int_C z^p \, \mathrm{d}z$$

we cannot use either the FTC or the Cauchy Integral Theorem. Hence, we must compute it as a contour integral using a parametrization of C. Let $z(t) = re^{it}$, $0 \le t \le 2\pi$, parametrize C oriented counterclockwise. Since $z'(t) = ire^{it}$, we find

$$\int_C z^p \, \mathrm{d}z = \int_0^{2\pi} (re^{it})^p \cdot ire^{it} \, \mathrm{d}t = ir^{p+1} \int_0^{2\pi} e^{i(p+1)t} \, \mathrm{d}t.$$

We now consider two subcases. If $p = -1$, then

$$\int_C z^p \, \mathrm{d}z = \int_C z^{-1} \, \mathrm{d}z = ir^{(-1)+1} \int_0^{2\pi} e^{i((-1)+1)t} \, \mathrm{d}t = i \int_0^{2\pi} \mathrm{d}t = 2\pi i.$$

If $p = -2, -3, \ldots$, then

$$\int_C z^p \, \mathrm{d}z = ir^{p+1} \int_0^{2\pi} e^{i(p+1)t} \, \mathrm{d}t = \frac{ir^{p+1}}{i(p+1)} e^{i(p+1)t} \bigg|_{t=0}^{t=2\pi} = \frac{r^{p+1}}{p+1} [e^{i(p+1)2\pi} - 1] = 0$$

since $e^{i(p+1)2\pi} = 1$ as $p \in \mathbb{Z}$.

In summary, we have

$$\int_C z^p \, \mathrm{d}z = \begin{cases} 2\pi i, & \text{if } p = -1, \\ 0, & \text{if } p \in \mathbb{Z},\ p \neq 0. \end{cases}$$

Example 19.2. Suppose that $C = \{|z| = r\}$ is the circle of radius $r > 0$ centred at 0 oriented counterclockwise. Let $a \in \mathbb{C}$. Compute

$$\int_C \frac{1}{z-a} \, \mathrm{d}z$$

assuming that (i) a is outside C and (ii) a is inside C. (Note that a is never *on* C.)

Solution. If a is outside C, then there is some simply connected domain D containing C but not a. Moreover, the function $f(z) = (z-a)^{-1}$ is analytic in D with $f'(z) = -(z-a)^{-2}$ for $z \in D$ so that $f'(z)$ is continuous in D. Hence, the hypotheses of the Cauchy Integral Theorem, Basic Version have been met so that

$$\int_C \frac{1}{z-a} \, \mathrm{d}z = 0.$$

On the other hand, suppose that a is inside C and let R denote the interior of C. Since the function $f(z) = (z-a)^{-1}$ is not analytic in any domain containing R, we cannot apply the Cauchy Integral Theorem. Hence, we must compute it as a contour integral using a parametrization of C. Let $z(t) = re^{it}$, $0 \leq t \leq 2\pi$, parametrize C oriented counterclockwise. Since $z'(t) = ire^{it}$, we find

$$\int_C \frac{1}{z-a} \, \mathrm{d}z = \int_0^{2\pi} \frac{1}{re^{it} - a} \cdot ire^{it} \, \mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{re^{it} - a} \, \mathrm{d}t.$$

Now we would like to compute the Riemann integral. Observe that if the imaginary unit i were absent (and assuming $a \in \mathbb{R}$) we would find

$$\int_0^{2\pi} \frac{re^t}{re^t - a} \, \mathrm{d}t = \log(re^t - a) \bigg|_{t=0}^{t=2\pi}$$

where the log is a natural logarithm. However, the inclusion of the imaginary unit i means that we cannot simply say

$$\int_0^{2\pi} \frac{ire^{it}}{re^{it} - a} \, \mathrm{d}t = \mathrm{Log}(re^{it} - a) \bigg|_{t=0}^{t=2\pi} \qquad (*)$$

where the Log is the principal value of the logarithm. In fact, observe that

$$\operatorname{Log}(re^{it}-a)\Big|_{t=0}^{t=2\pi} = \operatorname{Log}(re^{i2\pi}-a) - \operatorname{Log}(r-a) = \operatorname{Log}(r-a) - \operatorname{Log}(r-a) = 0$$

and so if $(*)$ were actually true, we would find

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{ire^{it}}{re^{it}-a}\,\mathrm{d}t = 0.$$

But we have already shown that in the case $a=0$, the integral does not equal 0, but rather

$$\int_C \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

This means that in order to compute

$$\int_0^{2\pi} \frac{ire^{it}}{re^{it}-a}\,\mathrm{d}t$$

we must consider the real and imaginary parts separately. Now,

$$\begin{aligned}
\frac{re^{it}}{re^{it}-a} = \frac{re^{it}}{re^{it}-a}\cdot\frac{re^{-it}-\bar{a}}{re^{-it}-\bar{a}} &= \frac{r^2-\bar{a}re^{it}}{r^2-\bar{a}re^{it}-are^{-it}+|a|^2} \\
&= \frac{r^2-|a|re^{i(t-\operatorname{Arg} a)}}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)} \\
&= \frac{r^2-|a|r\cos(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)} - i\frac{|a|r\sin(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)}
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^{2\pi} \frac{ire^{it}}{re^{it}-a}\,\mathrm{d}t \\
&\quad= \int_0^{2\pi} \frac{|a|r\sin(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)}\,\mathrm{d}t + i\int_0^{2\pi} \frac{r^2-|a|r\cos(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)}\,\mathrm{d}t.
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^{2\pi} \frac{|a|r\sin(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)} = \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a))\Big|_{t=0}^{t=2\pi} \\
&\quad= \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(2\pi-\operatorname{Arg} a)) - \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(0-\operatorname{Arg} a)) \\
&\quad= \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(\operatorname{Arg} a)) - \frac{1}{2}\log(r^2+|a|^2-2|a|r\cos(\operatorname{Arg} a)) \\
&\quad= 0
\end{aligned}$$

since $r>|a|$. (Recall that a is *inside* C. This is crucial in order for the logarithms to be natural logarithms of positive real numbers.) However, it is rather tricky to compute

$$\int_0^{2\pi} \frac{r^2-|a|r\cos(t-\operatorname{Arg} a)}{r^2+|a|^2-2|a|r\cos(t-\operatorname{Arg} a)}\,\mathrm{d}t = \int_{-\operatorname{Arg} a}^{2\pi-\operatorname{Arg} a} \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t.$$

One way of doing it is as follows. Observe that

$$\begin{aligned}
\int \frac{-2|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t &= \int \frac{r^2+|a|^2-2|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\
&= \int 1\,\mathrm{d}t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\
&= t - \int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t
\end{aligned}$$

so that

$$\int \frac{-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t = \frac{t}{2} - \frac{1}{2}\int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t$$

which in turn implies that

$$\begin{aligned}
\int \frac{r^2-|a|r\cos t}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t &= \int \frac{r^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t + \frac{t}{2} - \frac{1}{2}\int \frac{r^2+|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\
&= \frac{t}{2} + \frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t.
\end{aligned}$$

Recall that $\cos t = \cos^2(t/2) - \sin^2(t/2)$ and $1 = \cos^2(t/2) + \sin^2(t/2)$ so that

$$\begin{aligned}
&\frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r\cos t}\,\mathrm{d}t \\
&\quad= \frac{1}{2}\int \frac{r^2-|a|^2}{r^2+|a|^2-2|a|r(\cos^2(t/2)-\sin^2(t/2))}\,\mathrm{d}t \\
&\quad= \frac{1}{2}\int \frac{r^2-|a|^2}{(r^2+|a|^2)(\cos^2(t/2)+\sin^2(t/2)) - 2|a|r(\cos^2(t/2)-\sin^2(t/2))}\,\mathrm{d}t \\
&\quad= \frac{1}{2}\int \frac{r^2-|a|^2}{(r^2+|a|^2-2|a|r)\cos^2(t/2) + (r^2+|a|^2+2|a|r)\sin^2(t/2)}\,\mathrm{d}t \\
&\quad= \frac{1}{2}\int \frac{r^2-|a|^2}{(r-|a|)^2\cos^2(t/2) + (r+|a|)^2\sin^2(t/2)}\,\mathrm{d}t \\
&\quad= \frac{r^2-|a|^2}{2(r-|a|)^2}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t \\
&\quad= \frac{r+|a|}{2(r-|a|)}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t.
\end{aligned}$$

Make the substitution

$$\theta = \frac{r+|a|}{r-|a|}\tan(t/2) \quad \text{so that} \quad \mathrm{d}\theta = \frac{r+|a|}{2(r-|a|)}\sec^2(t/2)\,\mathrm{d}t$$

which implies

$$\frac{r+|a|}{2(r-|a|)}\int \frac{\sec^2(t/2)}{1+\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right)^2}\,\mathrm{d}t = \int \frac{1}{1+\theta^2}\,\mathrm{d}\theta = \arctan\theta = \arctan\left(\frac{r+|a|}{r-|a|}\tan(t/2)\right).$$

Hence,

$$\int \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t = \frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right).$$

However, if we want to compute the definite integral

$$\int_0^{2\pi} \frac{r^2 - |a| r \cos(t - \operatorname{Arg} a)}{r^2 + |a|^2 - 2|a| r \cos(t - \operatorname{Arg} a)} \,\mathrm{d}t = \int_{-\operatorname{Arg} a}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t$$

we cannot just write

$$\int_{-\operatorname{Arg} a}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t = \left[\frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right)\right]_{t=-\operatorname{Arg} a}^{t=2\pi - \operatorname{Arg} a}.$$

The reason for this is that the definite integral on the left in the previous expression is actually improper. This can be seen by considering the expression on the right. The trouble spot is when $t = \pi$; that is, $\tan(\pi/2)$ is not defined, and so we cannot just compute the integral over the range $-\operatorname{Arg} a \le t \le 2\pi - \operatorname{Arg} a$ without considering what happens when $t = \pi$. Since $\operatorname{Arg} a \in (-\pi, \pi]$, we can conclude that $-\operatorname{Arg} a \le \pi \le 2\pi - \operatorname{Arg} a$ so that the trouble spot is actually in the range of integration. That is,

$$\begin{aligned}
&\int_{-\operatorname{Arg} a}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t \\
&\qquad = \int_{-\operatorname{Arg} a}^{\pi} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t + \int_{\pi}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t \\
&\qquad = \lim_{\theta \uparrow \pi} \int_{-\operatorname{Arg} a}^{\theta} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t + \lim_{\theta \downarrow \pi} \int_{\theta}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t.
\end{aligned}$$

Now

$$\begin{aligned}
&\lim_{\theta \uparrow \pi} \int_{-\operatorname{Arg} a}^{\theta} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t \\
&\quad = \lim_{\theta \uparrow \pi} \left[\frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right)\right]_{t=-\operatorname{Arg} a}^{t=\theta} \\
&\quad = \frac{\pi}{2} + \frac{\operatorname{Arg} a}{2} - \arctan\left(\frac{r+|a|}{r-|a|} \tan\left(-\frac{\operatorname{Arg} a}{2}\right)\right) + \lim_{\theta \uparrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right) \qquad (*)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\theta \downarrow \pi} \int_{\theta}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \,\mathrm{d}t \\
&\quad = \lim_{\theta \downarrow \pi} \left[\frac{t}{2} + \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right)\right]_{t=\theta}^{t=2\pi - \operatorname{Arg} a} \\
&\quad = \frac{\pi}{2} - \frac{\operatorname{Arg} a}{2} + \arctan\left(\frac{r+|a|}{r-|a|} \tan\left(\pi - \frac{\operatorname{Arg} a}{2}\right)\right) - \lim_{\theta \downarrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right) \\
&\hspace{30em} (**)
\end{aligned}$$

so that adding $(*)$ and $(**)$ and using the fact that $\tan(-\theta) = \tan(\pi - \theta)$ implies

$$\int_{-\operatorname{Arg} a}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \, \mathrm{d}t$$
$$= \pi + \lim_{\theta \uparrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right) - \lim_{\theta \downarrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right)$$

Since $\tan(t/2) \to \infty$ as $t \uparrow \pi$ and since $\tan(t/2) \to -\infty$ as $t \downarrow \pi$, we conclude

$$\lim_{\theta \uparrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right) = \frac{\pi}{2}$$

and

$$\lim_{\theta \downarrow \pi} \arctan\left(\frac{r+|a|}{r-|a|} \tan(t/2)\right) = -\frac{\pi}{2}$$

so that

$$\int_{-\operatorname{Arg} a}^{2\pi - \operatorname{Arg} a} \frac{r^2 - |a| r \cos t}{r^2 + |a|^2 - 2|a| r \cos t} \, \mathrm{d}t = \pi + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi.$$

In summary, we have shown that

$$\int_C \frac{1}{z-a} \, \mathrm{d}z = \int_0^{2\pi} \frac{ire^{it}}{re^{it} - a} \, \mathrm{d}t = 2\pi i$$

if a is inside $C = \{|z| = r\}$, the circle of radius $r > 0$ centred at 0 oriented counterclockwise.

Mathematics 312 (Fall 2012) October 22, 2012
Prof. Michael Kozdron

Lecture #20: Applications of the Cauchy Integral Theorem

Last lecture we derived two results by direct calculation, namely

$$\int_C \frac{1}{z}\,\mathrm{d}z = 2\pi i$$

where C is the circle of radius $r > 0$ centred at 0 oriented counterclockwise and, more generally,

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

for any $a \in \mathbb{C}$ with $|a| < r$. Note that the first result is a special case of the second result (i.e., with $a = 0$). Also note that the first result was relatively easy to derive whereas the second result was not.

Example 20.1. Suppose that $C_a = \{|z-a| < r\}$ denotes the circle of radius $r > 0$ centred at a oriented counterclockwise. Compute

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z.$$

Solution. Since the function

$$f(z) = \frac{1}{z-a}$$

is not analytic at a which happens to be inside C_a, we must evaluate this contour integral by definition. Let $z(t) = a + re^{it}$, $0 \le t \le 2\pi$, parametrize C so that $z'(t) = ire^{it}$. Therefore,

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{z(t)-a} z'(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{ire^{it}}{a+re^{it}-a}\,\mathrm{d}t = \int_0^{2\pi} i\,\mathrm{d}t = 2\pi i.$$

We have now determined by direct calculations that

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

where C is the circle of radius $r > 0$ centred at 0 oriented counterclockwise, C_a is the circle of radius $r > 0$ centred at a oriented counterclockwise, and $|a| < r$. We will now show that it is easy to determine

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

as a consequence of the fact that

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

which will render our horrendous calculation from Lecture #19 unnecessary. Consider Figure 20.1 below.

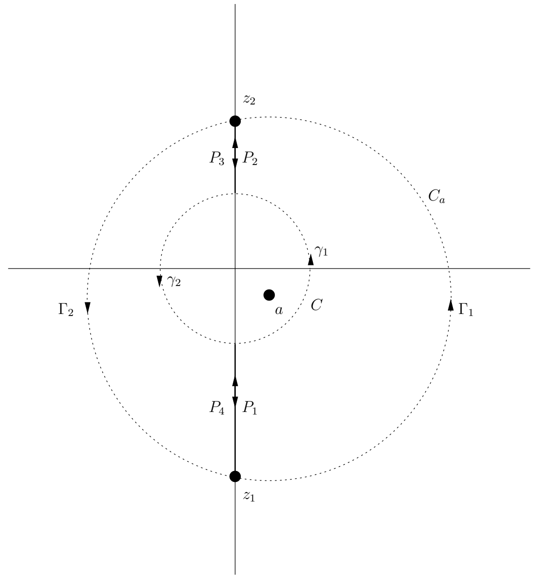

Figure 20.1: Continuous deformation of C into C_a.

Here we have taken z_1 and z_2 to be the points of intersection of C_a with the negative and positive imaginary axes, respectively. The curve Γ_1 connects z_1 to z_2 counterclockwise along C_a while the curve Γ_2 connects z_2 with z_1 counterclockwise along C_a. Note that

$$\int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = \int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Let P_1 be the curve that connects C_a to C along the negative imaginary axis, and let P_4 be the curve that connects C to C_a along the negative imaginary axis. Similarly, let P_2 be the curve that connects C to C_a along the positive imaginary axis, and let P_3 be the curve that connects C_a to C along the positive imaginary axis. Finally, let γ_1 be the curve counterclockwise along C connecting P_1 to P_2, and let γ_2 be the curve counterclockwise along C connecting P_3 to P_4. Note that

$$\int_{C} \frac{1}{z-a}\,\mathrm{d}z = \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Now here is the key. The function

$$f(z) = \frac{1}{z-a}$$

is analytic everywhere in $\mathbb{C}$ except at a. Therefore, the Fundamental Theorem of Calculus tells us that the value of the contour integral of $f(z)$ over any curve going from z_1 to z_2 is independent of the curve taken (as long as that curve does not pass through a). Now here are two curves going from z_1 to z_2, namely (i) Γ_1, and (ii) $P_1 \oplus \gamma_1 \oplus P_2$. This means

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_1 \oplus \gamma_1 \oplus P_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_2} \frac{1}{z-a}\,\mathrm{d}z.$$

Similarly,

$$\int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_3\oplus\gamma_2\oplus P_4} \frac{1}{z-a}\,\mathrm{d}z = \int_{P_3} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.$$

Adding these together gives

$$\begin{aligned}&\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z \\ &= \int_{P_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_3} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_2} \frac{1}{z-a}\,\mathrm{d}z + \int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.\end{aligned}$$

However, since P_1 and P_4 follow the same path but in different directions, we have

$$\int_{P_1} \frac{1}{z-a}\,\mathrm{d}z = -\int_{P_4} \frac{1}{z-a}\,\mathrm{d}z.$$

Similarly, P_2 and P_3 follow the same path but in the different directions so that

$$\int_{P_2} \frac{1}{z-a}\,\mathrm{d}z = -\int_{P_3} \frac{1}{z-a}\,\mathrm{d}z.$$

This implies

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z.$$

But we know

$$\int_{\Gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\Gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z$$

and

$$\int_{\gamma_1} \frac{1}{z-a}\,\mathrm{d}z + \int_{\gamma_2} \frac{1}{z-a}\,\mathrm{d}z = \int_{C} \frac{1}{z-a}\,\mathrm{d}z$$

so that

$$\int_{C} \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

as desired.

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Prof. Michael Kozdron

Lecture #21: Applications of the Cauchy Integral Theorem

Last lecture we showed that if C is the circle of radius $r > 0$ centred at 0 oriented counterclockwise, C_a is the circle of radius $r > 0$ centred at a oriented counterclockwise, and $|a| < r$, then

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = \int_{C_a} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i. \tag{$*$}$$

By constructing an appropriate picture, we were able to *continuously deform* C to C_a and show that $(*)$ held. Of course, the same construction holds for *any* contour C oriented counterclockwise surrounding the point a as shown in Figure 21.1 below.

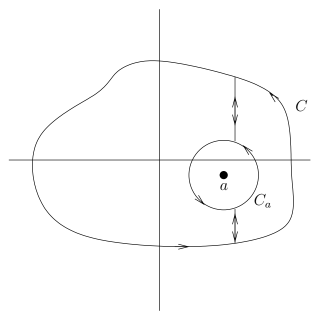

Figure 21.1: Continuous deformation of C into C_a.

This leads to the following fact.

Theorem 21.1. *If C is a closed contour in the complex plane oriented counterclockwise and $a \in \mathbb{C}$ is in the interior of C, then*

$$\int_C \frac{1}{z-a}\,\mathrm{d}z = 2\pi i.$$

Example 21.2. Compute

$$\int_C \frac{1}{z+i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Since $|-i| = 1 < 2$, we see that $a = -i$ is inside C so that

$$\int_C \frac{1}{z+i}\,\mathrm{d}z = 2\pi i.$$

Example 21.3. Compute

$$\int_C \frac{1}{2z+i}\,\mathrm{d}z$$

where $C = \{|z| = 2\}$ is the circle of radius 2 centred at 0 oriented counterclockwise.

Solution. Since the integrand is not of the form $(z-a)^{-1}$, we cannot use the fact immediately. However,

$$\int_C \frac{1}{2z+i}\,\mathrm{d}z = \frac{1}{2}\int_C \frac{1}{z+i/2}\,\mathrm{d}z = \frac{1}{2}(2\pi i) = \pi i$$

since $a=-i/2$ is inside of the circle of radius 2 centred at 0.

Example 21.4. Compute

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z$$

where C is the simple closed contour indicated in Figure 21.2 below.

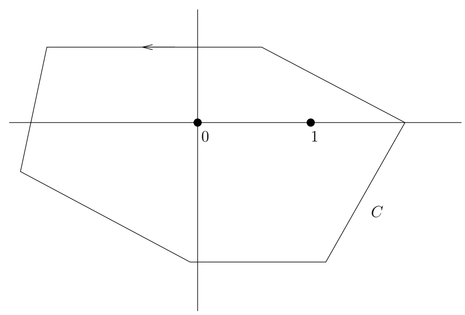

Figure 21.2: Figure for Example 21.4.

Solution. The trick is to use partial fractions on the integrand. That is,

$$\frac{3z-2}{z^2-z} = \frac{3z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

if and only if

$$A(z-1)+Bz = (A+B)z - A = 3z-2.$$

This, of course, is true if and only if $A=2$ and $B=1$. That is,

$$\frac{3z-2}{z^2-z} = \frac{2}{z} + \frac{1}{z-1}$$

and so

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = \int_C \frac{2}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2(2\pi i) + 2\pi i = 6\pi i.$$

Example 21.5. Compute

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z$$

where C is the simple closed contour indicated in Figure 21.3 below.

Solution. Again we can write

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z.$$

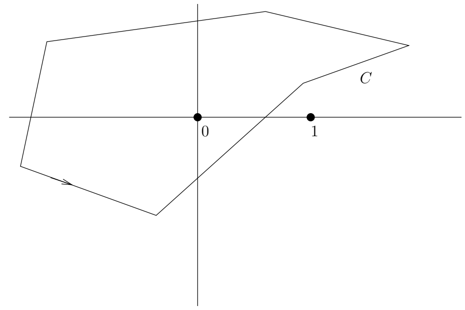

Figure 21.3: Figure for Example 21.5.

This time, however, $(z-1)^{-1}$ is analytic inside C since 1 is not inside C. The Cauchy Integral Theorem, Basic Version tells us that

$$\int_C \frac{1}{z-1}\,\mathrm{d}z = 0.$$

Therefore,

$$\int_C \frac{3z-2}{z^2-z}\,\mathrm{d}z = 2\int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z-1}\,\mathrm{d}z = 2(2\pi i) + 0 = 4\pi i.$$

Example 21.6. Compute

$$\int_C \frac{1}{z^2-1}\,\mathrm{d}z$$

where C is the simple closed contour indicated in Figure 21.4 below.

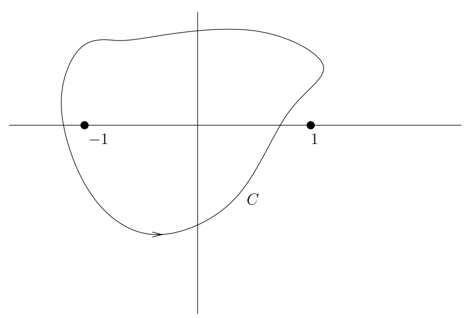

Figure 21.4: Figure for Example 21.6.

Solution. Using partial fractions, we find

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1/2}{z-1} - \frac{1/2}{z+1}.$$

Since $z=1$ is not inside C, the Cauchy Integral Theorem, Basic Version tells us that

$$\int_C \frac{1}{z-1}\,\mathrm{d}z = 0.$$

Therefore,

$$\int_C \frac{1}{z^2-1}\,\mathrm{d}z = \frac{1}{2}\int_C \frac{1}{z-1}\,\mathrm{d}z - \frac{1}{2}\int_C \frac{1}{z+1}\,\mathrm{d}z = 0 - \frac{1}{2}(2\pi i) = -\pi i.$$