Solutions to Math 305 Midterm Exam #2

1. We find

$$\left|\frac{4n^2+7}{3n^2-2n}-\frac{4}{3}\right| = \left|\frac{12n^2+21-12n^2+8n}{3(3n^2-2n)}\right| = \left|\frac{8n+21}{3(3n^2-2n)}\right|.$$

By the triangle inequality,

$$|8n+21| \le 8n+21n = 29n$$

for all $n \ge 1$. Moreover,

$$|3n^2-2n| = 3n^2-2n \ge 3n^2-2n^2 = n^2$$

provided that $3n^2-2n \ge 0$ and $-2n \ge -2n^2$. Note that if $n \ge 1$, then it is certainly true that $3n \ge 2$, or equivalently, $3n^2-2n \ge 0$. Moreover, if $n \ge 1$, then clearly $2n \le 2n^2$ so that $-2n \ge -2n^2$. In brief, if $n \ge 1$, then

$$\left|\frac{4n^2+7}{3n^2-2n}-\frac{4}{3}\right| \le \frac{|8n+21|}{3|3n^2-2n|} \le \frac{29n}{3n^2} = \frac{29}{3n}.$$

Thus, let $\varepsilon > 0$ be given. If

$$N = \frac{29}{3\varepsilon},$$

and $n \ge N$, then

$$\left|\frac{4n^2+7}{3n^2-2n}-\frac{4}{3}\right| \le \frac{29}{3n} \le \frac{29}{3N} = \epsilon$$

proving that

$$\lim_{n\to\infty}\frac{4n^2+7}{3n^2-2n} = \frac{4}{3}$$

as required.

2. Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is a Cauchy sequence, there exists an N_1 such that

$$|a_n - a_m| < \frac{\varepsilon}{2}$$

whenever $n, m \ge N_1$. Since $\{b_n\}$ is a Cauchy sequence, there exists an N_2 such that

$$|b_n - b_m| < \frac{\varepsilon}{2}$$

whenever $n, m \ge N_2$. Therefore, let $N = \max\{N_1, N_2\}$. If $n, m \ge N$, then

$$|c_n - c_m| = |(a_n - b_n) - (a_m - b_m)| \le |a_n - a_m| + |b_n - b_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so that $\{c_n\}$ is, in fact, a Cauchy sequence.

3. Since $\{b_n\}$ is bounded, there exists some M such that

$$|b_n| \le M$$

for all n. Therefore, let $\varepsilon > 0$ be given. Since $a_n \to 0$, there exists an N such that if $n \ge N$, then

$$|a_n - 0| = |a_n| \le \frac{\varepsilon}{M}.$$

Therefore, if $n \ge N$, then

$$|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| \le \frac{\varepsilon}{M} \cdot M = \varepsilon$$

proving that $a_n b_n \to 0$ as required.

4. (a) Suppose that $c_n = a_n + b_n$ for all n. Let $\mathcal{A}$ denote the set of subsequential limits of $\{a_n\}$, let $\mathcal{B}$ denote the set of subsequential limits of $\{b_n\}$, and let $\mathcal{C}$ denote the set of subsequential limits of $\{c_n\}$ so that

$$\limsup_{n\to\infty} a_n = \sup \mathcal{A} \in \mathbb{R}, \quad \limsup_{n\to\infty} b_n = \sup \mathcal{B} \in \mathbb{R}, \quad \text{and} \quad \limsup_{n\to\infty} c_n = \sup \mathcal{C} \in \mathbb{R}$$

using the facts that $\{a_n\}$ and $\{b_n\}$ are bounded sequences. Moreover, the facts that $\{a_n\}$ and $\{b_n\}$ are bounded sequences also imply that $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$. Thus, we need to show that

$$\sup \mathcal{C} \le \sup \mathcal{A} + \sup \mathcal{B}.$$

Suppose that c_{n_k} is a convergent subsequence of c_n with limit $c \in \mathcal{C}$. Since $c_{n_k} = a_{n_k} + b_{n_k}$, there are two possibilities. Either (i) a_{n_k} is a convergent subsequence of a_n with limit $a \in \mathcal{A}$ and b_{n_k} is a convergent subsequence of b_n with limit $b \in \mathcal{B}$ or (ii) a_{n_k} is not a convergent subsequence of a_n and b_{n_k} is not a convergent subsequence of b_n. (By the limit theorems, it is not possible for c_{n_k} to converge along with exactly one of a_{n_k} and b_{n_k}.) We will now consider the two cases separately. For the first case, suppose that $c_1 = \sup \mathcal{C}$ and assume that c_{n_k} is a convergent subsequence of c_n with limit c_1. Since $c_{n_k} = a_{n_k} + b_{n_k}$ with $c_{n_k} \to c_1$, $a_{n_k} \to a$, and $b_{n_k} \to b$, we conclude that

$$c_1 = a + b \le \sup \mathcal{A} + \sup \mathcal{B}$$

since $a \le \sup \mathcal{A}$ and $b \le \sup \mathcal{B}$. For the second case, suppose again that $c_1 = \sup \mathcal{C}$ and that c_{n_k} is a convergent subsequence of c_n with limit c_1. Since $c_{n_k} = a_{n_k} + b_{n_k}$, but neither a_{n_k} nor b_{n_k} converge, we need to consider further subsequences. Thus, let $a_{n_{k_j}}$ be a convergent subsequence of a_{n_k} with limit $a \in \mathcal{A}$. The subsequence $c_{n_{k_j}}$ of c_{n_k} necessarily converges to c_1 since $c_{n_k} \to c_1$, and so it follows that $b_{n_{k_j}}$ is a convergent subsequence of b_{n_k} with limit, say, $b \in \mathcal{B}$. Hence, we conclude as before that

$$c_1 = a + b \le \sup \mathcal{A} + \sup \mathcal{B}.$$

In either case, we have $\sup \mathcal{C} \le \sup \mathcal{A} + \sup \mathcal{B}$ which completes the proof.

4. **(b)** Suppose that $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$ so that $c_n = a_n + b_n = 0$ for all n. Note that

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1 \quad \text{whereas} \quad \limsup_{n\to\infty} c_n = 0$$

so that

$$0 = \limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 1 + 1 = 2.$$

5. Let $a \in \mathbb{R}$ be arbitrary. Note that

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2).$$

Therefore, if $|x-a| < 1$, then $|x| = |x-a+a| \le |x-a| + |a| < 1 + |a|$ so that

$$|x^2 + ax + a^2| \le |x|^2 + |a||x| + a^2 < (1+|a|)^2 + |a|(1+|a|) + a^2 = 1 + 3|a| + 3a^2.$$

Let $\epsilon > 0$ be given and choose

$$\delta = \min\left\{\frac{\varepsilon}{1+3|a|+3a^2}, 1\right\}.$$

This implies that if $|x-a| < \delta$, then

$$|x^3 - a^3| = |x-a||x^2 + ax + a^2| < \varepsilon$$

so that

$$\lim_{x\to a} x^3 = a^3$$

as required.

6. **(a)** Recall that f is continuous at c if and only if $f(x_n)$ converges to $f(c)$ for any sequence x_n converging to c. Suppose now that x_n converges to 2. Since $2 \in \mathbb{Q}$ and $f(2) = 10$, we must show that $f(x_n) \to 10$. Let $\varepsilon > 0$ and find N such that $n \ge N$ implies $|x_n - 2| < \varepsilon$. There are now two possibilities. If $x_n \in \mathbb{Q}$ and $n > N$, then $f(x_n) = 5x_n$ so that $|f(x_n) - 10| = |5x_n - 10| = 2|x_n - 5| < 2\epsilon$. If $x_n \in \mathbb{R}\setminus\mathbb{Q}$ and $n > N$, then $f(x_n) = x_n^2 + 6$ so that $|f(x_n) - 10| = |x_n^2 + 6 - 10| = |x_n^2 - 4| = |x_n - 2||x_n + 2|$. Since $|x_n - 2| < \varepsilon$ we know that $|x_n + 2| \le |x_n - 2| + 4 < \varepsilon + 4$ which implies that

$$|f(x_n) - 10| = |x_n - 2||x_n + 2| < \varepsilon(\varepsilon + 4).$$

In either case, if $n \ge N$, then we can make $|f(x_n) - 10|$ arbitrarily close to 0 proving that $f(x_n) \to f(2)$ whenever $x_n \to 2$.

6. **(b)** To show that f is discontinuous at 1, we must show that there exists a sequence $x_n \to 1$ for which $f(x_n)$ does not converge to $f(1)$. Suppose that $x_n \in \mathbb{R}\setminus\mathbb{Q}$ with $x_n \to 1$. As an example, take $x_n = 1 - (\sqrt{2}n)^{-1}$. Since $x_n \in \mathbb{R}\setminus\mathbb{Q}$, we know that

$$f(x_n) = x_n^2 + 6 \to 7.$$

However, since $1 \in \mathbb{Q}$, we know that $f(1) = 5$. Therefore, $f(x_n)$ does not converge to $f(1)$ proving that f is not continuous at 1.