Math 305 Midterm Exam #2 – November 15, 2011

This exam is worth 60 points.

This exam has 6 problems and 1 numbered page.

You have 75 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For problems with multiple parts, all parts are equally weighted.*

You may cite without proof results that were proved in class. Unless otherwise specified, you may use any technique to solve a particular problem.

This exam is closed book. There are no aids whatsoever allowed.

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Some Notation

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers.
- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ denotes the set of integers.
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\right\}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.

All sequences on this exam are sequences of real numbers indexed by natural numbers.

1. (*10 points*) Use the ε-N definition of a limit to prove directly that

$$\lim_{n\to\infty} \frac{4n^2+7}{3n^2-2n} = \frac{4}{3}.$$

2. (*10 points*) Suppose that $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ are Cauchy sequences. Define the sequence $\{c_1, c_2, \ldots\}$ by setting $c_n = a_n - b_n$. Prove that $\{c_1, c_2, \ldots\}$ is a Cauchy sequence.

3. (*10 points*) Suppose that $\{a_1, a_2, \ldots\}$ is a sequence with $a_n \to 0$. Suppose further that $\{b_1, b_2, \ldots\}$ is a bounded sequence. Prove that

$$\lim_{n\to\infty} (a_n b_n) = 0.$$

4. (*10 points*)

Suppose that $\{a_n\}$ and $\{b_n\}$ are bounded sequences.

(a) Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

(b) Find a specific example to show that equality need not hold in **(a)**.

5. (*10 points*) Use the ε-δ definition of a limit to prove directly that

$$\lim_{x\to a} x^3 = a^3$$

where $a \in \mathbb{R}$ is arbitrary.

6. (*10 points*) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 5x, & \text{if } x \in \mathbb{Q}, \\ x^2 + 6, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(a) Prove that f is continuous at $c = 2$.

(b) Prove that f is discontinuous at $c = 1$.