Math 305 Fall 2011
Open and Closed Sets

Definition. Suppose that $S \subseteq \mathbb{R}$ is a set. We say that a point $x \in S$ is an *interior point* of S if there exists some $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$. We write $\operatorname{int} S$ to denote the set of all interior points of S. We say that a point $x \in S$ is a *boundary point* of S if for every $\varepsilon > 0$ both $N(x;\varepsilon) \cap S \neq \emptyset$ and $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$. We write $\operatorname{bd} S$ to denote the set of all boundary points of S. We say that S is *closed* if $\operatorname{bd} S \subseteq S$ and we say that S is *open* if $\operatorname{bd} S \subseteq (\mathbb{R} \setminus S)$.

Theorem. *The set $S \subseteq \mathbb{R}$ is open if and only if $S = \operatorname{int} S$.*

Proof. Suppose that $S = \operatorname{int} S$. This implies that for every $x \in S$, there exists an $\varepsilon > 0$ (the ε may depend on the particular x chosen) such that $N(x;\varepsilon) \subseteq S$. Therefore, it must be the case for this x and the associated ε that $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) = \emptyset$. Therefore, x cannot be a boundary point of S; that is, $x \notin \operatorname{bd} S$. In other words, we have shown that $S \cap (\operatorname{bd} S) = \emptyset$ so that the only possible points in $\operatorname{bd} S$ come from points not in S; that is, $\operatorname{bd} S \subseteq (\mathbb{R} \setminus S)$. Hence, by definition, S is open.

On the other hand, suppose that S is open. This means that $\operatorname{bd} S \subseteq (\mathbb{R} \setminus S)$ or, equivalently, $\operatorname{bd} S \cap S = \emptyset$. Thus, if $x \in S$, then there must exist an $\varepsilon > 0$ such that $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) = \emptyset$ (for otherwise x *would be* a boundary point). Hence, $N(x;\varepsilon) \subseteq S$ so that $x \in \operatorname{int} S$. As this is true for every $x \in S$, it must be the case that $S \subseteq \operatorname{int} S$. Since it is always the case that $\operatorname{int} S \subseteq S$, we conclude that $S = \operatorname{int} S$.

Since we have shown both implications, the proof is complete. ◻

As a result of the definition of open set and the previous theorem, we have a useful way of determining whether or not a set is open. It is simply a restatement of what $S = \operatorname{int} S$ means.

Corollary. *The set S is open if and only if for every $x \in S$, there exists an $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$.*

We can use this corollary to prove the following result.

Theorem. *If $\{E_\alpha : \alpha \in I\}$ is an arbitrary collection of open sets, then $E = \bigcup_{\alpha \in I} E_\alpha$ is open.*

Proof. To prove that E is open, we will show that for every $x \in E$, there exists an $\varepsilon > 0$ such that $N(X;\varepsilon) \subseteq E$. Let $x \in E$. Since E is the union of $\{E_\alpha : \alpha \in I\}$, it must be the case that there exists some $\alpha_0 \in I$ such that $x \in E_{\alpha_0}$. The fact that E_{α_0} is open means that there is some $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq E_{\alpha_0}$. (We are actually using the previous corollary at this step.) Since $N(x;\varepsilon) \subseteq E_{\alpha_0}$, it is necessarily the case that

$$N(x;\varepsilon) \subseteq \bigcup_{\alpha \in I} E_\alpha.$$

Hence, we have shown that for an arbitrary $x \in E$, there is some ε-neighbourhood of X that is contained in E; that is, E is open by the previous corollary. ◻