Math 305 Fall 2011
Solutions to Assignment #3

Exercises 12.3 and 12.4. Suppose that S is the subset of $\mathbb{R}$ given in each problem.

(a) $\sup S = 3$, $\max S = 3$, $\inf S = 1$, $\min S = 1$

(b) $\sup S = \pi$, $\max S = \pi$, $\inf S = 3$, $\min S = 3$

(c) $\sup S = 4$, $\max S = 4$, $\inf S = 0$, $\min S = 0$

(d) $\sup S = 4$, $\inf S = 0$ ($\max S$ and $\min S$ do not exist)

(e) $\sup S = 1$, $\max S = 1$, $\inf S = 0$ ($\min S$ does not exist)

(f) $\sup S = 1$, $\inf S = 0$, $\min S = 0$ ($\max S$ does not exist)

(g) $\sup S = 1$, $\inf S = 1/2$, $\min S = 1/2$ ($\max S$ does not exist)

(h) $\sup S = 3/2$, $\max S = 3/2$, $\inf S = -2$, $\min S = -2$

(i) $\inf S = 0$, $\min S = 0$ ($\sup S$ and $\max S$ do not exist)

(j) $\sup S = 4$ ($\max S$, $\inf S$, and $\min S$ do not exist)

(k) $\sup S = 1$, $\max S = 1$, $\inf S = 1$, $\min S = 1$

(l) $\sup S = 2$, $\inf S = 0$ ($\max S$ and $\min S$ do not exist)

(m) $\sup S = 5$ ($\max S$, $\inf S$, and $\min S$ do not exist)

(n) $\sup S = \sqrt{5}$, $\inf S = -\sqrt{5}$ ($\max S$ and $\min S$ do not exist)

Exercise 12.5. Suppose that S is a nonempty bounded subset of $\mathbb{R}$ and let $m = \sup S$. In order to prove that $m \in S$ if and only if $m = \max S$, we must prove two implications. First of all, suppose that $m \in S$. We will show that this implies that $m = \max S$. By the definition of supremum, we know that $m \geq s$ for all $s \in S$. This means that m is an upper bound for S. Since we have assumed that $m \in S$, the definition of maximum tells us that $m = \max S$. On the other hand, suppose that $m = \max S$. We will show that this implies $m \in S$. Since $m = \max S$, we know by the definition of maximum that m is an upper bound for S and is a member of S. That is, $m \in S$. Having established both implications, the proof is complete.

Exercise 12.9 (a). Although this problem seems straightforward, the trick is to realize that we need to use an axiom for the natural numbers to solve it. Suppose that $y > 0$ is fixed and consider the set

$$S = \{m \in \mathbb{N} : m > y\}.$$

We would like to use the well-ordering property to conclude that $\min S$ exists since S is bounded below. The key, though, is to realize that the well-ordering property only applies if S is both nonempty and bounded below. Therefore, we need to use the fact (equivalent to the Archimedean property) that for any $y > 0$, there exists an element $m_0 \in \mathbb{N}$ such that $m_0 > y$. This element m_0 belongs to S so that S is nonempty and therefore $\min S$ exists. Suppose that we denote $n_0 = \min S$ so that by the definition of minimum, $n_0 \in S$. The fact that $n_0 \in S$ means that $n_0 > y$. The fact that it is the minimal element means that $n_0 - 1$ is *not* in S. Therefore,

$n_0 - 1 \in S^c = \{m \in N : m \leq y\}$ so that $n_0 - 1 \leq y$. Hence, combining these two inequalities implies that

$$n_0 - 1 \leq y < y$$

and the proof is complete.

Exercise 12.9 (b). In order to establish that the n in part (a) is unique, we will use a proof by contradiction. That is, suppose to the contrary that the n in part (a) is not unique. This means that there exists $n \in \mathbb{N}$ such that $n - 1 \leq y < n$ and there exists $m \in \mathbb{N}$ such that $m - 1 \leq y < m$. Since m and n are assumed to differ, one will be less than the other. Assume without loss of generality that $m < n$. Therefore, since m and n are natural numbers, it must be the case that $m + 1 \leq n$, or equivalently, $m \leq n - 1$. Observe that this gives us three inequalities, namely

(i) $m - 1 \leq y < m$,

(ii) $n - 1 \leq y < n$,

(iii) $m \leq n - 1$.

Notice that **(ii)** and **(iii)** combined tell us that $m \leq n - 1 \leq y < n$; that is, $m \leq y$. However, **(i)** tells us that $y < m$. However, it is not possible for $y > 0$ to satisfy $m \leq y < m$. This must contradict the assumption that there exists $m \neq n$ satisfying part (a).

As a remark, this is not the only way to combine inequalities to derive a contradiction. We can re-write the inequalities in the equivalent forms

(i) $m \leq y + 1 < m + 1$,

(ii) $n \leq y + 1 < n + 1$,

(iii) $m < n$.

If we now take **(iii)** and combine it with **(ii)**, then we find $m < n \leq y + 1$. If we now use the fact from **(i)** that $y + 1 < m + 1$, then we have the chain of inequalities

$$m < n < m + 1.$$

Subtracting m from each part shows this is equivalent to

$$0 < n - m < 1.$$

However, $n - m$ is necessarily an integer and since there are no integers strictly between 0 and 1, we have arrived at a contradiction to the assumption that there exists $m \neq n$ satisfying part (a).