Math 261 Midterm Exam – November 9, 2011

This exam is worth 50 points.

This exam has 1 problem with 5 parts on 1 numbered page. All parts are equally weighted.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Use the exam booklet for all of your written work, but not for your computer work. Indicate just your final answers on the answer sheet. You do not need to submit any computer code.

This exam is open book. You are permitted to use your class notes and textbook.

Instructor: Michael Kozdron

Consider the function $f(x) = \sqrt{1 + \log(1 + x)}$ defined for $0 \leq x \leq 1$. This function does not have a simple antiderivative and so it must be analyzed numerically.

(a) Numerically determine a midpoint Riemann sum approximation for

$$\int_0^1 \sqrt{1 + \log(1 + x)}\, \mathrm{d}x$$

using $n = 5000$ subintervals.

(b) Find the second degree Taylor polynomial about the point $y_0 = 0$ for $g(y) = \sqrt{1 + y}$. In other words, you will find explicit values of a_0, a_1, and a_2 such that

$$\sqrt{1 + y} \approx a_0 + a_1 y + a_2 y^2.$$

(c) Find the second degree Taylor polynomial about the point $x_0 = 0$ for $h(x) = \log(1 + x)$. In other words, you will find explicit values of b_0, b_1, and b_2 such that

$$\log(1 + x) \approx b_0 + b_1 x + b_2 x^2.$$

(d) Notice that $f(x) = g(h(x))$. Therefore, substitute you answer from **(c)** into your answer for **(b)** to determine a portion of the Taylor expansion for $\sqrt{1 + \log(1 + x)}$. That is, let $y = \log(1 + x)$ so that

$$\begin{aligned} \sqrt{1 + y} = \sqrt{1 + \log(1 + x)} &\approx a_0 + a_1[\log(1 + x)] + a_2[\log(1 + x)]^2 \\ &\approx a_0 + a_1[b_0 + b_1 x + b_2 x^2] + a_2[b_0 + b_1 x + b_2 x^2]^2. \end{aligned}$$

Now ignore the powers of three and higher to determine the second degree approximation to $\sqrt{1 + \log(1 + x)}$.

(e) Consider the second degree approximation to $\sqrt{1 + \log(1 + x)}$ that you found in **(d)**, namely

$$\sqrt{1 + \log(1 + x)} \approx c_0 + c_1 x + c_2 x^2$$

where c_0, c_1 and c_2 are what you found in **(d)**. You will now determine the value of z such that

$$0.6 = \int_0^z (c_0 + c_1 x + c_2 x^2)\, \mathrm{d}x$$

as follows. Compute (by hand) the integral

$$\int_0^z (c_0 + c_1 x + c_2 x^2)\, \mathrm{d}x.$$

This will be a polynomial in z. Set that polynomial equal to 0.6 and use the method of false position to numerically determine z with a tolerance of 10^{-6}.

Answers to Math 261 Midterm Exam

Name: ______________________________

(a) $\boxed{}$

(b) $\boxed{}$

(c) $\boxed{}$

(d) $\boxed{}$

(e) $\boxed{}$

Solutions to Math 261 Midterm Exam

(a) Using OCTAVE, we find

```
> format long
> n = 5000;
> z = [1/(2*n):1/n:(n-1)/n+1/(2*n)];
> MPS = sum(sqrt(1+log(1+z))/n)
MPS = 1.17432617262822.
```

(b) If $g(y) = \sqrt{1+y}$, then

$$g'(y) = \frac{1}{2}(1+y)^{-1/2} \quad \text{and} \quad g''(y) = -\frac{1}{4}(1+y)^{-3/2}$$

so that $g(0) = 1$, $g'(0) = \dfrac{1}{2}$, and $g''(0) = -\dfrac{1}{4}$. Therefore, the second degree Taylor polynomial about the point $y = 0$ for $g(y) = \sqrt{1+y}$ is

$$g(0) + g'(0)y + \frac{g''(0)}{2}y^2 = 1 + \frac{y}{2} - \frac{y^2}{8}.$$

(c) If $h(x) = \log(1+x)$, then

$$h'(x) = \frac{1}{1+x} \quad \text{and} \quad h''(x) = -\frac{1}{(1+x)^2}$$

so that $h(0) = 0$, $h'(0) = 1$, $h''(0) = 1$. Therefore, the second degree Taylor polynomial about the point $x = 0$ for $h(x) = \dfrac{1}{1+x}$ is

$$h(0) + h'(0)x + \frac{h''(0)}{2}x^2 = x - \frac{x^2}{2}.$$

(d) From our work in **(b)** and **(c)**, we find

$$\begin{aligned}\sqrt{1+\log(1+x)} &\approx 1 + \frac{\log(1+x)}{2} - \frac{[\log(1+x)]^2}{8} \approx 1 + \frac{x - \frac{x^2}{2}}{2} - \frac{[x - \frac{x^2}{2}]^2}{8} \\ &= 1 + \frac{x}{2} - \frac{x^2}{4} - \frac{x^2}{8} + \frac{x^3}{8} - \frac{x^4}{32} \approx 1 + \frac{x}{2} - \frac{3x^2}{8}.\end{aligned}$$

(e) We find

$$0.6 = \int_0^z \left(1 + \frac{x}{2} - \frac{3x^2}{8}\right)\,\mathrm{d}x = \left(x + \frac{x^2}{4} - \frac{x^3}{8}\right)\Bigg|_0^z = z + \frac{z^2}{4} - \frac{z^3}{8}$$

so that z satisfies

$$0.6 = z + \frac{z^2}{4} - \frac{z^3}{8} \quad \text{or, equivalently,} \quad 24 = 40z + 10z^2 - 5z^3.$$

Using OCTAVE with the **False Position Method Algorithm** from Assignment #4 gives

```
> falsepos(0,1,1E-6,1000)
ans = 0.545842763436761.
```