Stat 252 Winter 2016
Assignment #7

This assignment is due at the beginning of class on Monday, April 11, 2016. Your solutions will be graded based on both correctness *and* exposition. In particular, neatness and grammar count. You must write out solutions using full sentences (including capital letters to start sentences and periods to end them) and no abbreviations. That is, symbols such as $\therefore$ and $\Rightarrow$ are forbidden; write out the full words *therefore* and *implies* in their place.

1. Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed with density function

$$f(y|\theta) = \theta \exp(-\theta y)$$

where $y > 0$ and $\theta > 0$. As usual, let $\overline{Y}$ denote the sample mean given by

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

It is known that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = 1 / \overline{Y}$.

(a) Consider testing $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$. Verify that the rejection region for the generalized likelihood ratio test of these hypotheses is of the form

$$\left\{\overline{Y} \exp(-\theta_0 \overline{Y}) \leq C\right\}$$

for some suitable constant C.

(b) Suppose, to be specific, that $\theta_0 = 1$, and that a random sample of size $n = 10$ is conducted. If the observed data yield $\overline{y} = 1.25$, perform the generalized likelihood ratio test at the approximate significance level $\alpha = 0.10$.

2. Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed with density function

$$f(y|\theta) = \theta^2 \, y \exp(-\theta y)$$

where $y > 0$ and $\theta > 0$. As usual, let $\overline{Y}$ denote the sample mean given by

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

It is known that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = 2 / \overline{Y}$.

(a) Consider testing $H_0 : \theta = 1$ against $H_A : \theta \neq 1$. Verify that the generalized likelihood ratio for this hypothesis testing problem is

$$\Lambda = \left(\frac{\overline{Y}}{2}\right)^{2n} \exp(2n - n\overline{Y}).$$

(b) Suppose that a random sample of size $n = 5$ is conducted and the observed data yield $\overline{y} = 1.0$. Perform the generalized likelihood ratio test at the approximate significance level $\alpha = 0.05$.

3. Suppose that $Y \sim \text{Bin}(n, \theta)$ where θ follows a $\beta(a, b)$ prior density. That is, assume

$$g(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}, \quad 0 \leq \theta \leq 1.$$

Determine the posterior distribution of θ given $Y = y$.

4. Recall that a random variable Y is said to be Poisson with parameter θ if it has density function

$$f(y|\theta) = \frac{\theta^y e^{-\theta}}{y!}, \quad y = 0, 1, 2, \ldots.$$

Suppose that the random variables $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Poisson(θ) where the parameter θ is unknown. If the prior distribution of θ is $\Gamma(\alpha, \beta)$, determine the posterior distribution of θ given $\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n\}$.

5. Suppose that the random variable Y has a Raleigh(θ) distribution where $\theta > 0$ is an unknown parameter so that

$$f(y|\theta) = \frac{2}{\theta} y e^{-y^2/\theta}, \quad y > 0.$$

Suppose further that the prior for θ is given by

$$g(\theta) = \frac{2}{\sqrt{\pi}} e^{-\theta^2}, \quad \theta > 0.$$

(a) Determine an expression for the posterior density function $f(\theta|y)$.

(b) Determine the posterior density function $f(\theta|y = 1)$ by numerically estimating the normalizing constant.