Statistics 252 Midterm #1 – February 5, 2007

This exam has 5 problems and is worth 40 points. **Instructor:** Michael Kozdron

You must answer all of the questions in the exam booklet provided.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

This exam is closed-book, except that one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is permitted. No other aids are allowed.*

The following facts might prove useful.

- If the random variable Y has the Gamma(α, β) distribution, then
$$f_Y(y) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-y/\beta}, \quad y > 0.$$
Furthermore,
$$m_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}, \quad E(Y) = \alpha\beta, \quad \mathrm{Var}(Y) = \alpha\beta^2.$$

- If the random variable Y has the Exp(λ) distribution, then
$$f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda}, \quad y > 0.$$
Furthermore,
$$m_Y(t) = \frac{1}{1-\lambda t}, \quad E(Y) = \lambda, \quad \mathrm{Var}(Y) = \lambda^2.$$

- If the random variable Y has the $\mathcal{N}(\mu, \sigma^2)$ distribution, then
$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < y < \infty.$$
Furthermore,
$$m_Y(t) = \exp\left\{\mu t + \frac{t^2\sigma^2}{2}\right\}, \quad E(Y) = \mu, \quad \mathrm{Var}(Y) = \sigma^2.$$

1. (*10 points*) Suppose that $Y_1, \ldots, Y_n$ is a random sample where the density of each random variable Y_i is

$$f_Y(y) = 2\theta^2 y^{-3}, \quad y \geq \theta$$

for some parameter $\theta > 1$. Let $\hat{\theta} := \min\{Y_1, \ldots, Y_n\}$.

(a) Calculate $B(\hat{\theta})$, the bias of $\hat{\theta}$.

(b) Determine the value of the constant c for which $\hat{\theta}_1 := c\hat{\theta}$ is unbiased.

(c) Calculate $\mathrm{MSE}(\hat{\theta}_1)$, the mean-square error of $\hat{\theta}_1$.

2. (*6 points*) An electrical circuit consists of four batteries connected in parallel to a lightbulb. We model the battery lifetimes X_1, X_2, X_3, X_4 as independent and identically distributed Uniform$(0, \theta)$ random variables (where $\theta > 0$ is a parameter). Our experiment to measure the lifetime of the lightbulb Y is stopped when the last battery fails. Determine the density function for the random variable $Y := \max\{X_1, X_2, X_3, X_4\}$.

3. (*10 points*) Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed Exp(θ) random variables.

(a) Let $X_1 := Y_1 + \cdots + Y_n$. Use moment generating functions to show that X_1 has a Gamma(n, θ) distribution.

(b) Let $X_2 := \min\{Y_1, \ldots, Y_n\}$. Show that X_2 has an Exp$\left(\frac{\theta}{n}\right)$ distribution.

(c) Verify that $\hat{\theta}_1 := \dfrac{X_1}{n}$ is an unbiased estimator for θ.

(d) Verify that $\hat{\theta}_2 := nX_2$ is an unbiased estimator for θ.

(e) Which estimator, $\hat{\theta}_1$ or $\hat{\theta}_2$, is preferred for the estimation of θ?

4. (*8 points*) Let $X_1, \ldots, X_n$ be i.i.d. $\mathcal{N}(\mu_1, \sigma_1^2)$ and let $Y_1, \ldots, Y_m$ be i.i.d. $\mathcal{N}(\mu_2, \sigma_2^2)$. Suppose further that X_i and Y_j are independent, $i = 1, \ldots, n$; $j = 1, \ldots, m$.

(a) Determine the sampling distribution of $\overline{X} - \overline{Y}$.

(b) Suppose that σ_1 and σ_2 are known, but that μ_1 and μ_2 are unknown. Use your result of **(a)** to construct a $1 - \alpha$ confidence interval for $\mu_1 - \mu_2$. (Of course, $0 < \alpha < 1$.)

5. (*6 points*)

(a) Suppose that you have been asked to analyze a data set, and you are planning to use a particular estimator to estimate a parameter. In the context of Stat 252, explain why it is important to determine the sampling distribution of this estimator.

(b) In the context of Stat 252, explain the multiple meanings of the term *statistic*.