Stat 252 Winter 2007
The Standard Normal Distribution Function

Suppose that $Y \sim \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known, and that we are interested in constructing a confidence interval for θ based on Y. In a previous lecture, we solved this problem by recognizing that

$$Z = \frac{Y-\theta}{\sigma} \sim \mathcal{N}(0,1)$$

which enabled us to use a table to find the critical value $z_{\alpha/2}$ such that $1-\alpha = P(-z_{\alpha/2} \le Z \le z_{\alpha/2})$. We then substituted back yielding

$$1-\alpha = P\left(-z_{\alpha/2} \le \frac{Y-\theta}{\sigma} \le z_{\alpha/2}\right) = P\left(Y - z_{\alpha/2}\sigma \le \theta \le Y + z_{\alpha/2}\sigma\right)$$

so that $\left[Y - z_{\alpha/2}\sigma\, ,\, Y + z_{\alpha/2}\sigma\right]$ is the required $1-\alpha$ confidence interval.

Question. How does this construction relate to the pivotal method?

Answer. As the following example shows, this method is, in fact, the pivotal method!

Example. If $Y \sim \mathcal{N}(\theta, \sigma^2)$, then the density of Y is given by

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\theta)^2}{2\sigma^2}\right\}, \quad -\infty < y < \infty,$$

and so the distribution function of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\} dx.$$

As you know from your calculus classes, this integral cannot be evaluated in closed form. However, that does not stop us! If we let $U = \frac{Y-\theta}{\sigma}$, then

$$F_U(u) = P(U \le u) = P\left(Y \le u\sigma + \theta\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{u\sigma+\theta} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\} dx.$$

We now take a derivative with respect to U to find the density of U, namely

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{[(u\sigma+\theta)-\theta]^2}{2\sigma^2}\right\} \cdot \frac{d}{du}(u\sigma+\theta) \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$$

provided that $-\infty < u < \infty$. We now recognize this as the density of a $\mathcal{N}(0,1)$ random variable. In order to construct a symmetric $1-\alpha$ confidence interval, we must find a and b such that

$$P(a \le U \le b) = 1-\alpha, \quad P(U < a) = P(U > b) = \frac{\alpha}{2}.$$

That is, we must now find a and b such that

$$\int_a^b f_U(u)\, du = 1-\alpha, \quad \int_{-\infty}^{a} f_U(u)\, du = \int_b^{\infty} f_U(u)\, du = \frac{\alpha}{2}.$$

However, the question arises: Exactly how do we find a such that

$$\int_{-\infty}^{a} f_U(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{u^2}{2}}\, du = \frac{\alpha}{2}?$$

Since we cannot perform the integration symbolically, the answer, of course, is to use a table! Alternatively, we can use a computer algebra system such as MAPLE which performs numerical integration. For instance, if $\alpha = 0.05$, then we find

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1.96} e^{-\frac{u^2}{2}}\, du \approx 0.025 \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{1.96}^{\infty} e^{-\frac{u^2}{2}}\, du \approx 0.025.$$

It is now worth noting that we are interested in finding a and b which satisfy

$$F(a) = \int_{-\infty}^{a} f_U(u)\,du = \frac{\alpha}{2} \quad \text{and} \quad 1 - F(b) = \int_{b}^{\infty} f_U(u)\,du = \frac{\alpha}{2}$$

for a given α. In other words, we need to find a and b such that

$$a = F^{-1}(\alpha/2) \quad \text{and} \quad b = F^{-1}(1 - \alpha/2).$$

The question, therefore, of whether or not we can use the pivotal method explicitly is equivalent to the question of whether or not we can invert the distribution function. As we have seen in the previous example, this is not possible for the normal distribution. It is for this reason that normal tables need to be constructed. Fortunately, extremely efficient algorithms exist for determining normal values, and so with modern computers a large number of decimal places of accuracy can be achieved.

When implementing such algorithms, the symmetry of the normal curve is heavily used. Suppose that $Z \sim \mathcal{N}(0,1)$. Let

$$\phi(z) := \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

denote the density function of Z, and let

$$\Phi(z) := \int_{-\infty}^{z} \phi(x)\,dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

denote the distribution function of Z. The next exercise asks you to prove one symmetry relationship, and the exercise following asks you to express the distribution function for a $\mathcal{N}(\mu, \sigma^2)$ random variable in terms of Φ.

Exercise 1. Show that if $\Phi(z)$ denotes the standard normal distribution function, then $1 - \Phi(z) = \Phi(-z)$.

Exercise 2. Show that if $X \sim \mathcal{N}(\mu, \sigma)$, then the distribution function of X is given by

$$F_X(x) = \Phi\left(\frac{x - \mu}{\sigma}\right).$$

In numerical analysis, the so-called *error function* is used instead of Φ. It is defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-x^2}\,dx.$$

Exercise 3. Show that if $-\infty < z < \infty$, then $\operatorname{erf}(z) = 2\,\Phi(\sqrt{2}\,z) - 1$.

The final two exercises are extremely important in the application of probability and statistics to finance. In fact, these exercises ask you to prove special cases of the Black-Scholes formula. The general Black-Scholes formula for pricing options has had a profound impact on the world of finance. Indeed, this is now considered basic knowledge for anyone studying modern economics. In 1997, Myron Scholes and Robert Merton were awarded the Nobel Prize in Economics for this work. (Fischer Black died in 1995.)

Notation. We write $x^+ := \max\{0, x\}$ to denote the *positive part* of x.

Exercise 4. Suppose that $Z \sim \mathcal{N}(0,1)$, and let $K > 0$ be a constant. Show that

$$\mathbb{E}(\ (e^Z - K)^+\ ) = e^{1/2}\,\Phi(1 - \log K) - K\Phi(-\log K).$$

Exercise 5. Suppose that $Z \sim \mathcal{N}(0,1)$, and let a, $b > 0$, and $K > 0$ be constants. Show that

$$\mathbb{E}(\ (ae^{bZ} - K)^+\ ) = e^{b^2/2}\,\Phi\left(b + \frac{1}{b}\log\frac{a}{K}\right) - \frac{K}{a}\Phi\left(\frac{1}{b}\log\frac{a}{K}\right).$$