Stat 252 Winter 2007
The Incomplete Gamma Function

As we saw in the previous lecture with the normal distribution, it was not possible to "invert the standard normal distribution function" and solve $F(a) = \alpha/2$ in general. The solution, instead, was to use a table to look up a for a fixed choice of α. Because of the ubiquity of the normal distribution, and because of the existence of efficient algorithms, normal tables are readily available. There are, in fact, other important distributions with this feature. For instance, tables exist for both the t distribution and the χ^2 distribution. Again, the reason that such tables are available is because of the importance of these distributions in practice, and because of the relative computational ease with which such tables can be produced. One other example is the gamma distribution. Although the gamma distribution is widely used in practice (including actuarial science), it is difficult to compile gamma tables. One area of current research in computer science is to develop efficient algorithms for calculating critical values associated with the gamma distribution. It is, therefore, worth studying this problem in some detail.

Example. Suppose that the random variable Y has density function

$$f_Y(y) = \frac{\theta^{-252}}{251!}\, y^{251}\, e^{-y/\theta}, \quad y > 0$$

where $\theta > 0$ is a parameter. Use the pivotal method to construct a symmetric confidence interval for θ based on Y with coverage probability $1-\alpha$ when $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$.

Solution. We begin by observing that $Y \sim \text{Gamma}(252, \theta)$. The distribution function of Y is given by

$$F_Y(y) = \frac{\theta^{-252}}{251!} \int_0^y x^{251}\, e^{-x/\theta}\, dx$$

which we also discover cannot be readily evaluated in closed form. If we let $U = \frac{Y}{\theta}$, then

$$F_U(u) = P(U \le u) = P\left(Y \le u\theta\right) = \frac{\theta^{-252}}{251!} \int_0^{u\theta} x^{251}\, e^{-x/\theta}\, dx.$$

We now take a derivative with respect to U to find the density of U, namely

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{\theta^{-252}}{251!} (u\theta)^{251}\, e^{-u\theta/\theta} \cdot \frac{d}{du}(u\theta) = \frac{1}{251!} u^{251}\, e^{-u}$$

provided that $u > 0$. We recognize this as the density of a Gamma$(252, 1)$ random variable. In order to construct a symmetric $1-\alpha$ confidence interval, we must find a and b such that

$$P(a \le U \le b) = 1 - \alpha, \quad P(U < a) = P(U > b) = \frac{\alpha}{2}.$$

That is, we must now find a and b such that

$$\int_a^b f_U(u)\, du = 1 - \alpha, \quad \int_0^a f_U(u)\, du = \int_b^\infty f_U(u)\, du = \frac{\alpha}{2}.$$

- Suppose that $\alpha = 0.10$. We must find a and b such that

  $$\frac{1}{251!} \int_0^a u^{251}\, e^{-u}\, du = 0.05 \quad \text{and} \quad \frac{1}{251!} \int_b^\infty u^{251}\, e^{-u}\, du = 0.05.$$

  Using MAPLE, we compute $a \approx 226.47$ and $b \approx 278.65$.

- Suppose that $\alpha = 0.05$. We must find a and b such that

$$\frac{1}{251!}\int_0^a u^{251}\, e^{-u}\, du = 0.025 \quad \text{and} \quad \frac{1}{251!}\int_b^\infty u^{251}\, e^{-u}\, du = 0.025.$$

  Using MAPLE, we compute $a \approx 221.85$ and $b \approx 284.05$.

- Suppose that $\alpha = 0.01$. We must find a and b such that

$$\frac{1}{251!}\int_0^a u^{251}\, e^{-u}\, du = 0.005 \quad \text{and} \quad \frac{1}{251!}\int_b^\infty u^{251}\, e^{-u}\, du = 0.005.$$

  Using MAPLE, we compute $a \approx 212.99$ and $b \approx 294.76$.

In all three cases, a and b satisfy

$$P\left(a \le \frac{Y}{\theta} \le b\right) = 1 - \alpha$$

and so the required $1 - \alpha$ confidence intervals for θ are therefore

$$\left[\frac{Y}{b}, \frac{Y}{a}\right].$$

The distribution function of a gamma random variable is often referred to as an incomplete gamma function. There are a variety of inconsistent definitions of the incomplete gamma function that one encounters in practice. Our definition is as follows.

Definition. If $p > 0$ and $y > 0$, let

$$\Gamma(p; y) := \int_0^y x^{p-1}\, e^{-x}\, dx$$

denote the *incomplete Gamma function*, and let

$$\Gamma(p) := \Gamma(p; \infty) := \int_0^\infty x^{p-1}\, e^{-x}\, dx$$

denote the *Gamma function.* Furthermore, let

$$G(p; y) := \int_y^\infty x^{p-1}\, e^{-x}\, dx$$

so that

$$\Gamma(p; y) + G(p; y) = \Gamma(p).$$

Notice that if the random variable Y has the Gamma(a, θ) distribution, then the density of Y is

$$f_Y(y) = \frac{\theta^{-a}}{\Gamma(a)} y^{a-1}\, e^{-y/\theta}\, dx$$

so that the distribution function of Y is

$$F_Y(y) = \frac{\theta^{-a}}{\Gamma(a)} \int_0^y x^{a-1}\, e^{-x/\theta}\, dx.$$

Example. Suppose that $\theta = 1$ so that $Y \sim \text{Gamma}(a, 1)$. In this case we can write $F_Y(y)$ in terms of the incomplete Gamma function as follows:

$$F_Y(y) = \frac{1}{\Gamma(a)} \int_0^y x^{a-1}\, e^{-x}\, dx = \frac{\Gamma(a; y)}{\Gamma(a)}.$$

Example. In general, suppose that $Y \sim \text{Gamma}(a, \theta)$ so that

$$F_Y(y) = \frac{\theta^{-a}}{\Gamma(a)} \int_0^y x^{a-1}\, e^{-x/\theta}\, dx.$$

Let $u = x/\theta$ so that $\theta du = dx$ and

$$F_Y(y) = \frac{\theta^{-a}}{\Gamma(a)} \int_0^{y/\theta} (u\theta)^{a-1}\, e^{-u}\, \theta du = \frac{1}{\Gamma(a)} \int_0^{y/\theta} u^{a-1}\, e^{-u}\, du.$$

In other words,

$$F_Y(y) = \frac{\Gamma(a; y/\theta)}{\Gamma(a)}.$$

It is not easy (even for a powerful computer) to determine explicitly values of the incomplete Gamma function. One approach that works reasonably well for integer values of a is to calculate the value of $\Gamma(1; y)$ and then use one of the following recursive formulæ to calculate $\Gamma(a; y)$ for other values of a. Unfortunately, this approach does not work very well if the value of a that we are interested in is not an integer.

Exercise 1. Use integration by parts to show that

$$\Gamma(a; y) = (a-1)\Gamma(a-1; y) - e^{-y} y^{a-1}. \qquad (*)$$

Deduce from $(*)$ that

$$\Gamma(a; y) = \frac{1}{a}\left[\Gamma(a+1; y) + e^{-y} y^{a}\right].$$

Example. Compute $\Gamma(4; y)$ using the recursive formula $(*)$. We begin by noticing that

$$\Gamma(1; y) = \int_0^y x^{1-1}\, e^{-x}\, dx = \int_0^y e^{-x}\, dx = 1 - e^{-y}.$$

Therefore,

- $\Gamma(2; y) = (2-1)\Gamma(2-1; y) - e^{-y} y^{2-1} = 1 - e^{-y} - ye^{-y}$,
- $\Gamma(3; y) = (3-1)\Gamma(3-1; y) - e^{-y} y^{3-1} = 2 - 2e^{-y} - 2ye^{-y} - y^2 e^{-y}$,
- $\Gamma(4; y) = (4-1)\Gamma(4-1; y) - e^{-y} y^{4-1} = 6 - 6e^{-y} - 6ye^{-y} - 3y^2 e^{-y} - y^3 e^{-y}$.

Remark. Integration by parts three times can also be used to show that

$$\int_0^y x^3\, e^{-x}\, dx = 6 - 6e^{-y} - 6ye^{-y} - 3y^2 e^{-y} - y^3 e^{-y}.$$

There are also recursive formulæ for $G(a; y)$. In fact, the following appear in the booklet provided to actuarial students writing Exam C/4 of the Society of Actuaries.

Exercise 2. Show that

$$G(a; y) = (a-1)G(a-1; y) + e^{-y} y^{a-1}. \qquad (**)$$

Deduce from $(**)$ that

$$G(a; y) = \frac{1}{a}\left[G(a+1; y) - e^{-y} y^{a}\right].$$