Final Exam – April 21, 2006

Problem 1. Suppose that $Y_1, \ldots, Y_n$ are independent and identically distributed random variables with each Y_i having density function

$$f_Y(y|\theta) = \frac{y^2}{2\,\theta^3} \exp\{-y/\theta\}, \quad y > 0,$$

where $\theta > 0$ is a parameter. It is known that $\mathbb{E}(Y_i) = 3\theta$ and $\mathrm{Var}(Y_i) = 3\theta^2$ for each $i = 1, \ldots, n$.

(a) Determine $\hat{\theta}_{\mathrm{MOM}}$, the method of moments estimator of θ.

(b) Compute the likelihood function $L(\theta)$ for this random sample.

(c) Show that the maximum likelihood estimator of θ is $\hat{\theta}_{\mathrm{MLE}} = \dfrac{1}{3n}\displaystyle\sum_{i=1}^{n} Y_i = \dfrac{\overline{Y}}{3}$.

(d) Based on your answers to **(a)** and **(c)**, show that both $\hat{\theta}_{\mathrm{MOM}}$ and $\hat{\theta}_{\mathrm{MLE}}$ are unbiased estimators of θ.

(e) Based on your answers to **(a)** and **(c)**, determine the relative efficiency of $\hat{\theta}_{\mathrm{MOM}}$ to $\hat{\theta}_{\mathrm{MLE}}$. Which do you prefer for the estimation of θ?

(f) Show that $\displaystyle\sum_{i=1}^{n} Y_i$ is a sufficient statistic for the estimation of θ.

(g) Explain why $\hat{\theta}_{\mathrm{MLE}}$ must also be a sufficient statistic for the estimation of θ.

(h) Find the Fisher information $I(\theta)$ in a single observation from this density.

(i) Carefully explain why $\hat{\theta}_{\mathrm{MLE}}$ must be the minimum variance unbiased estimator of θ.

(j) Using the standard approximation for the distribution of a maximum likelihood estimator based on the Fisher information, construct an approximate 90% confidence interval for θ.

(k) Determine the generalized likelihood ratio test statistic for the test of the hypothesis $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$.

To answer **(l)** and **(m)** below, suppose that a sample of size $n = 27$ produces $\displaystyle\sum_{i=1}^{27} y_i = 108$.

(l) Based on your confidence interval constructed in **(j)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?

(m) Using your generalized likelihood ratio test statistic constructed in **(k)** and on the above data, can you reject the hypothesis $H_0 : \theta = 1$ in favour of $H_A : \theta \neq 1$ at the significance level $\alpha = 0.10$?

(continued)

Problem 2. Certain resistors to be used in an electrical circuit are known to have resistances which are normally distributed with mean 200 ohms and standard deviation 10 ohms. These same resistors are also known to have lifetimes which are exponentially distributed with mean 12 hours. That is, if X denotes the resistance (in ohms) of a randomly selected resistor, then the density of X is

$$f_X(x) = \frac{1}{10 \cdot \sqrt{2\pi}} \exp\left\{-\frac{(x-200)^2}{2 \cdot 10^2}\right\} \quad \text{for } -\infty < x < \infty,$$

and if Y denotes the lifetime (in hours) of a randomly selected resistor, then the density of Y is

$$f_Y(y) = \frac{1}{12} e^{-y/12} \quad \text{for } y > 0.$$

Suppose that 4 resistors are selected at random to be used in an electrical circuit. If the resistors are connected in series, then there are two ways that the electrical circuit can fail. The first way is if the total resistance in the circuit exceeds 840 ohms. (Recall that when resistors are connected in series, the total resistance is simply the sum of the individual resistances.) The second way is when any one of the individual resistors fails.

(a) Compute the probability that the total resistance of these 4 resistors does *not* exceed 840 ohms. *(Hint: Compute the probability that the sum of the resistances for these 4 resistors is less than 840 ohms.)*

(b) Compute the probability that the circuit does *not* fail within the first 15 hours of operation. *(Hint: Compute the probability that the minimum lifetime for these 4 resistors is greater than 15 hours.)*

Problem 3. Suppose that $Y_1, \ldots, Y_9$ are independent and identically distributed $\mathcal{N}(\mu, 4)$ random variables, where the parameter μ is unknown. As usual, let

$$\overline{Y} = \frac{1}{9}\sum_{i=1}^{9} Y_i \quad \text{and} \quad S^2 = \frac{1}{8}\sum_{i=1}^{9}(Y_i - \overline{Y})^2.$$

You want to test $H_0 : \mu = 0$ against $H_A : \mu > 0$ by rejecting H_0 when $\overline{Y} > 1.55$. *(Note: You may consider your calculations accurate to two decimal places when consulting the appropriate table(s).)*

(a) Determine the significance level α of this test.

(b) Determine the power of this test when $\mu = 1$.

Problem 4. Let $Y_1, Y_2, \ldots, Y_n$ be independent Uniform$(0, \theta)$ random variables. Recall that a random variable with this distribution has density function

$$f_Y(y|\theta) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 < y < \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(continued)

(a) Compute $\hat{\theta}_{\text{MOM}}$, the method of moments estimator of θ.

(b) Determine the mean-squared error of $\hat{\theta}_{\text{MOM}}$.

(c) From previous work we know that the maximum likelihood estimator of θ is $\hat{\theta}_{\text{MLE}} = \max\{Y_1, \ldots, Y_n\}$. Determine the mean-squared error of $\hat{\theta}_{\text{MLE}}$. *(Hint: As a starting point, you might try to get an expression for the density function of* $\hat{\theta}_{\text{MLE}}$*.)*

Problem 5. When designing a hypothesis test, one of the first decisions that needs to be made is which hypothesis to assign to H_0 and which hypothesis to assign to H_A. One criterion that we discussed in class was to assign the hypothesis to H_0 for which the consequences of a type I error are worse than the consequences of a type II error. Consider the following hypothetical scenario. Suppose that Michael is convicted of a crime. If the judge determines that Michael is guilty of the crime, then Michael will be sent to prison. In this scenario, there are two possible hypothesis tests that can be considered.

- **Hypothesis Test A**: H_0: Michael is guilty vs. H_A: Michael is not guilty.
- **Hypothesis Test B**: H_0: Michael is not guilty vs. H_A: Michael is guilty.

In your opinion, for which of these two hypothesis tests (Test A or Test B) are the consequences of a type I error worse than the consequences of a type II error? Be sure to justify your decision, and to explain the practical significance (in terms of Michael being sent to prison) of the hypothesis test you have chosen.

Problem 6. Decide whether each of the following statements about hypothesis testing is either true (**T**) or false (**F**). Clearly circle your choice. You do *not* need to justify your answers.

T or **F** The significance level of a hypothesis test is equal to the probability of a type I error.

T or **F** If a test is rejected at significance level α, then the probability that the null hypothesis is true equals α.

T or **F** The probability that the null hypothesis is falsely rejected is equal to the power of the test.

T or **F** A type II error occurs when the test statistic falls in the rejection region of the test.

T or **F** The power of a test is determined by the null distribution of the test statistic.

T or **F** If the p-value is 0.03, then the corresponding test will reject at the significance level 0.02.

T or **F** If a test rejects at significance level 0.06, then the p-value is less than or equal to 0.06.

T or **F** The p-value of a test is the probability that the null hypothesis is correct.

(continued)

Problem 7. Assume that the outcome of an experiment is a single random variable Y, and that Y will be used as an estimator of an unknown parameter θ. The rejection region of the significance level $\alpha = 0.10$ hypothesis test

$$H_0 : \theta = 3 \quad \text{vs.} \quad H_A : \theta \neq 3$$

is $\text{RR} = \{Y > 7 \;\text{ or }\; Y < 2\}$. Based on this information, construct a 90% confidence interval for θ. *(Hint: Use the confidence interval–hypothesis test duality.)*

Problem 8. Consider a random variable Y with density function

$$f_Y(y|\theta) = \frac{2y}{\theta^2}\exp\{-y^2/\theta^2\}, \quad y > 0,$$

where $\theta > 0$ is a parameter. Using the pivotal quantity Y/θ, verify that

$$\left[\frac{Y}{\sqrt{-\log(\alpha_2)}}, \frac{Y}{\sqrt{-\log(1-\alpha_1)}}\right]$$

is a confidence interval for θ with coverage probability $1 - (\alpha_1 + \alpha_2)$.