Stat 252.01 Winter 2007
Assignment #7 Solutions

Important Remark: The factorizations of L into $L = g \cdot h$ are *not* unique. Many answers are possible.

Important Remark: Any one-to-one function of a sufficient statistic for θ is also sufficient for θ.

(9.30) If $Y_1, \ldots, Y_n$ are iid $\mathcal{N}(\mu, \sigma^2)$ random variables each with density

$$f_Y(y|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\},$$

then the likelihood function is

$$L(\mu, \sigma^2) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2\right\}.$$

(a) If μ is unknown, and σ^2 is known, then with

$$u = \overline{y}, \quad g(u, \mu) = \exp\left\{\frac{1}{2\sigma^2}\left(2\mu n u - \mu^2\right)\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2\right\},$$

the Factorization Theorem implies $\overline{Y}$ is sufficient for μ.

(b) If μ is known, and σ^2 is unknown, then with

$$u = \sum_{i=1}^{n}(y_i-\mu)^2, \quad g(u, \sigma^2) = (\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}u\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2},$$

the Factorization Theorem implies $\displaystyle\sum_{i=1}^{n}(Y_i-\mu)^2$ is sufficient for σ^2.

(c) If both μ and σ^2 are unknown, then with

$$u = (u_1, u_2) = \left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} y_i^2\right),$$

$$g(u, (\mu, \sigma^2)) = g((u_1, u_2), (\mu, \sigma^2)) = (\sigma^2)^{-n/2} \exp\left\{\frac{1}{2\sigma^2}\left(2\mu u_1 + u_2 - \mu^2\right)\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2},$$

the Factorization Theorem implies $\displaystyle\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i^2\right)$ is jointly sufficient for (μ, σ^2).

(9.34) If $Y_1, \dots, Y_n$ are iid geometric random variables each with density

$$f_Y(y|p) = p(1-p)^y,$$

for $y = 1, 2, 3, \dots$, then the likelihood function is

$$L(p) = p(1-p)^{\sum_{i=1}^n y_i}.$$

If

$$u = \overline{y}, \quad g(u,p) = p(1-p)^{nu}, \quad \text{and} \quad h(y_1, \dots, y_n) = 1,$$

then since $L(p) = g(u,p) \cdot h(y_1, \dots, y_n)$, we conclude by the Factorization Theorem that $\overline{Y}$ is sufficient for p.

(9.36) If $Y_1, \dots, Y_n$ are iid each with density

$$f_Y(y|\alpha, \beta) = \alpha \beta^\alpha y^{-(\alpha+1)}$$

for $y \geq \beta$, then for fixed β the likelihood function is

$$L(\alpha) = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^n y_i \right)^{-(\alpha+1)}.$$

If

$$u = \prod_{i=1}^n y_i, \quad g(u,\alpha) = \alpha^n \beta^{n\alpha} u^{-(\alpha+1)}, \quad \text{and} \quad h(y_1, \dots, y_n) = 1,$$

then since $L(\alpha) = g(u,\alpha) \cdot h(y_1, \dots, y_n)$, we conclude by the Factorization Theorem that $\prod_{i=1}^n Y_i$ is sufficient for α.

(9.74) (a) If $Y_1, \dots, Y_n$ are a random sample from the density function

$$f_Y(y|\theta) = \frac{1}{\theta} r y^{r-1} e^{-y^r/\theta}, \quad y > 0$$

where $\theta > 0$ is a parameter, then the likelihood function is

$$L(\theta) = \prod_{i=1}^n f_Y(y_i|\theta) = \prod_{i=1}^n \frac{1}{\theta} r y_i^{r-1} e^{-y_i^r/\theta} = \theta^{-n} \cdot r^n \cdot \left(\prod_{i=1}^n y_i \right)^{r-1} \cdot \exp\left(-\frac{1}{\theta} \sum_{i=1}^n y_i^r \right).$$

If

$$u = \sum_{i=1}^n y_i^r, \quad g(u,\theta) = \theta^{-n} \cdot \exp\left(-\frac{u}{\theta} \right), \quad h(y_1, \dots, y_n) = r^n \cdot \left(\prod_{i=1}^n y_i \right)^{r-1},$$

then the Factorization Theorem implies $\sum_{i=1}^n Y_i^r$ is sufficient for θ.

(c) Since the MLE obtained in **(b)**, namely

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n Y_i^r,$$

is a (one-to-one function of the) sufficient statistic from **(a)**, we conclude that if it is unbiased, or can be adjusted to be unbiased, then the MVUE of θ will be obtained. Since the Y_i are iid, we find

$$E(\hat{\theta}_{\text{MLE}}) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i^r) = E(Y_1^r) = \int_0^\infty y^r\, f_Y(y|\theta)\, dy = \frac{r}{\theta}\int_0^\infty y^{2r-1}\, e^{-y^r/\theta}\, dy.$$

To compute this integral, we use the substitution $x = y^r$, $dx = ry^{r-1}dr$ so that

$$E(\hat{\theta}_{\text{MLE}}) = \frac{r}{\theta}\int_0^\infty y^{2r-1}\, e^{-y^r/\theta}\, dy = \int_0^\infty \frac{x}{\theta} e^{-x/\theta}\, dx = \theta$$

since we recognize the last integral as the mean of a Gamma$(1, \theta)$ random variable. Therefore,

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} Y_i^r$$

is the MVUE of θ.

(9.75) **(b)** Since $Y_1, \ldots, Y_n$ are i.i.d. Uniform$(0, 2\theta + 1)$ random variables, then the variance of the underlying distribution is

$$\frac{(2\theta+1)^2}{12}.$$

In part **(a)** we determined that the maximum likelihood estimator of θ is

$$\hat{\theta}_{\text{MLE}} = \frac{\max\{Y_1, \ldots, Y_n\} - 1}{2}.$$

Therefore, the required MLE is

$$\frac{(2\hat{\theta}_{\text{MLE}}+1)^2}{12} = \frac{(2\frac{\max\{Y_1,\ldots,Y_n\}-1}{2}+1)^2}{12} = \frac{(\max\{Y_1, \ldots, Y_n\})^2}{12}.$$

(9.80) If $Y_1, \ldots, Y_n$ are i.i.d. with common density

$$f_Y(y|\theta) = (\theta+1)y^\theta, \quad 0 < y < 1$$

where $\theta > -1$ is a parameter, then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i|\theta) = \prod_{i=1}^{n}(\theta+1)y_i^\theta = (\theta+1)^n \left(\prod_{i=1}^{n} y_i\right)^\theta.$$

The maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ is obtained by maximizing $L(\theta)$, or equivalently, by maximizing the log-likelihood function $\ell(\theta)$ given by

$$\ell(\theta) = n\log(\theta+1) + \theta\sum_{i=1}^{n}\log y_i.$$

Since

$$\ell'(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^{n}\log y_i$$

we find $\ell'(\theta) = 0$ when

$$\frac{n}{\theta+1} + \sum_{i=1}^{n} \log y_i = 0 \quad \text{or} \quad \theta = -\frac{n}{\sum_{i=1}^{n} \log y_i} - 1.$$

Finally

$$\ell''(\theta) = -\frac{n}{(\theta+1)^2} < 0$$

so that by the second derivative test we conclude,

$$\hat{\theta}_{\text{MLE}} = -\frac{n}{\sum_{i=1}^{n} \log Y_i} - 1.$$

Recall that in Exercise **(9.61)** we found

$$\hat{\theta}_{\text{MOM}} = \frac{2\overline{Y} - 1}{1 - \overline{Y}}.$$

(9.81) If a coin is tossed twice, then there are three possibilities for the number of heads, namely 0,1, or 2. If the probability of flipping heads is p, then

$$P(Y = 0) = (1-p)^2, \quad P(Y = 1) = 2p(1-p), \quad P(Y = 2) = p^2.$$

It is a simple matter of plugging in the two possible values of p, namely 1/4 and 3/4 to determine that

- $p = 1/4$ maximizes $P(Y = 0)$,
- both $p = 1/4$ and $p = 3/4$ maximize $P(Y = 1)$,
- $p = 3/4$ maximizes $P(Y = 2)$.

Since the maximum likelihood estimator of p is simply the value that maximizes the likelihood function, we begin by determining the likelihood function. By definition, the likelihood function $L(p)$ is the product of the densities of the random variables in the sample. Since there is only one random variable being observed, we find that $L(p)$ is simply the density of Y, namely

$$P(Y = 0) = (1-p)^2, \quad P(Y = 1) = 2p(1-p), \quad P(Y = 2) = p^2.$$

Thus, the MLE depends on the observed value y so that

- if $y = 0$, then $\hat{p}_{\text{MLE}} = 1/4$,
- if $y = 1$, then $\hat{p}_{\text{MLE}} = 1/4$ and $\hat{p}_{\text{MLE}} = 3/4$ (there is no unique maximum),
- if $y = 2$, then $\hat{p}_{\text{MLE}} = 3/4$.