Statistics 252 Winter 2006 Midterm #2 – Solutions

1. (a) Since

$$\log f(y|\theta) = \log(\theta+1) + \theta\log(y)$$

we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{1}{\theta+1} + \log(y) \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{1}{(\theta+1)^2}.$$

Thus,

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{1}{(\theta+1)^2}.$$

(b) To find $\hat{\theta}_{\text{MOM}}$ we solve the equation $E(Y) = \overline{Y}$ for θ. Since

$$E(Y) = \int_0^1 y f(y|\theta)\,dy = (\theta+1)\int_0^1 y^{\theta+1}\,dy = \left(\frac{\theta+1}{\theta+2}\right) y^{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}$$

we conclude that

$$\frac{\theta+1}{\theta+2} = \overline{Y}.$$

Solving for θ yields

$$\hat{\theta}_{\text{MOM}} = \frac{2\overline{Y}-1}{1-\overline{Y}}.$$

2. Let $U = Y/\theta$ so that for $0 \le u \le 1$,

$$P(U \le u) = P(Y \le \theta u) = \int_0^{\theta u} 2\theta^{-2} y\,dy = \frac{y^2}{\theta^2}\Big|_0^{\theta u} = u^2.$$

The density function of U is therefore $f_U(u) = 2u$ for $0 \le u \le 1$. Thus, we must find a and b so that

$$\int_0^a 2u\,du = \frac{\alpha}{2} \quad \text{and} \quad \int_b^1 2u\,du = \frac{\alpha}{2}.$$

Computing the integrals we find $a^2 = \alpha/2$ and $1 - b^2 = \alpha/2$. Hence,

$$1-\alpha = P(a \le U \le b) = P\left(\sqrt{\alpha/2} \le \frac{Y}{\theta} \le \sqrt{1-\alpha/2}\right) = P\left(\frac{Y}{\sqrt{1-\alpha/2}} \le \theta \le \frac{Y}{\sqrt{\alpha/2}}\right).$$

In other words,

$$\left(\frac{Y}{\sqrt{1-\alpha/2}}, \frac{Y}{\sqrt{\alpha/2}}\right)$$

is a confidence interval for θ with coverage probability $1-\alpha$.

3. (a) Since

$$\log f(y|\theta) = y\log(\theta) - y - \log(y!)$$

we find

$$\frac{\partial}{\partial\theta}\log f(y|\theta) = \frac{y}{\theta} \quad \text{and} \quad \frac{\partial^2}{\partial\theta^2}\log f(y|\theta) = -\frac{y}{\theta^2}.$$

Thus,

$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2}\log f(Y|\theta)\right) = \frac{E(Y)}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}.$$

(b) If $Y \sim \text{Poisson}(\theta)$, then since $E(Y) = \theta$ we find that setting $E(Y) = \overline{Y}$ gives

$$\hat{\theta}_{\text{MOM}} = \overline{Y}.$$

(c) Since $E(Y_1) = \theta$, we conclude that

$$E(\hat{\theta}_{\text{MOM}}) = E(\overline{Y}) = E\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = E(Y_1) = \theta$$

so that $\hat{\theta}_{\text{MOM}}$ is an unbiased estimator of θ.

(d) Since $\text{Var}(Y_1) = \theta$, and since the Y_i are iid, we conclude

$$\text{Var}(\hat{\theta}_{\text{MOM}}) = \text{Var}(\overline{Y}) = \text{Var}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\text{Var}(Y_1)}{n} = \frac{\theta}{n}.$$

(e) The Cramer-Rao inequality tells us that an unbiased estimator $\hat{\theta}$ of θ must satisfy

$$\text{Var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} = \frac{\theta}{n}$$

since we found in **(a)** that $I(\theta) = 1/\theta$. From **(c)** we know that $\hat{\theta}_{\text{MOM}}$ is unbiased, and from **(d)** we know that

$$\text{Var}(\hat{\theta}_{\text{MOM}}) = \frac{\theta}{n}.$$

Hence, we have found an unbiased estimator, namely $\hat{\theta}_{\text{MOM}}$, whose variance attains the lower bound of the Cramer-Rao inequality. Hence, $\hat{\theta}_{\text{MOM}}$ must be the MVUE of θ.

4. (a) We see that since the Y_i are iid,

$$\text{Var}(\hat{p}) = \text{Var}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\text{Var}(Y_1)}{n} = \frac{p(1-p)}{n}.$$

Let $g(p) = p(1-p)/n$ so that $g'(p) = (1-2p)/n$. Therefore, we conclude that $g'(p) = 0$ when $1 - 2p = 0$ or when $p = 1/2$. Since $g''(p) = -2/n < 0$, the second derivative test implies that $p = 1/2$ is the global maximum for g. Thus, the maximum value of $\text{Var}(\hat{p})$ occurs when $p = 1/2$.

(b) Suppose that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ. The relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$, is defined as

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_1)}{\text{Var}(\hat{\theta}_2)}.$$

Therefore, if $\text{eff}(\hat{\theta}_1, \hat{\theta}_2) < 1$, then we conclude that $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$ so that $\hat{\theta}_1$ is the preferred estimator. Conversely, if $\text{eff}(\hat{\theta}_1, \hat{\theta}_2) > 1$, then we conclude that $\text{Var}(\hat{\theta}_1) > \text{Var}(\hat{\theta}_2)$ so that in this case $\hat{\theta}_2$ is preferred.