Stat 252.01 Winter 2006
Assignment #7 Solutions

(9.73) If $Y_1, \dots, Y_n$ are iid exponential(θ), then each Y_i has common density

$$f(y|\theta) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0.$$

Therefore, the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-y_i/\theta} = \theta^{-n} \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} y_i\right).$$

The maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ is obtained by maximizing $L(\theta)$, or equivalently, by maximizing the log-likelihood function $\ell(\theta)$ given by

$$\ell(\theta) = -n\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} y_i.$$

Since

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} y_i$$

we find $\ell'(\theta) = 0$ when

$$-\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} y_i = 0 \quad \text{or} \quad \theta = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Finally

$$\ell''(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} y_i$$

so that

$$\ell''\left(\frac{1}{n}\sum y_i\right) = -\frac{n^3}{\left(\sum y_i\right)^2} < 0.$$

By the second derivative test we conclude,

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \overline{Y}.$$

Since $\theta > 0$, we see that the function $T(\theta) = \theta^2$ is one-to-one. Therefore, the maximum likelihood estimator of θ^2 is

$$\hat{\theta}^2_{\text{MLE}} = \overline{Y}^2.$$

(9.74) (b) If $Y_1, \dots, Y_n$ are a random sample from the density function

$$f(y|\theta) = \frac{1}{\theta} r y^{r-1} e^{-y^r/\theta}, \quad y > 0$$

where $\theta > 0$ is a parameter, then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} r y_i^{r-1} e^{-y_i^r/\theta} = \theta^{-n} \cdot r^n \cdot \left(\prod_{i=1}^{n} y_i\right)^{r-1} \cdot \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} y_i^r\right).$$

The maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ is obtained by maximizing $L(\theta)$, or equivalently, by maximizing the log-likelihood function $\ell(\theta)$ given by

$$\ell(\theta) = -n \log \theta + n \log r + (r-1) \sum_{i=1}^{n} \log y_i - \frac{1}{\theta} \sum_{i=1}^{n} y_i^r.$$

Since

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} y_i^r$$

we find $\ell'(\theta) = 0$ when

$$-\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} y_i^r = 0 \quad \text{or} \quad \theta = \frac{1}{n} \sum_{i=1}^{n} y_i^r.$$

Finally

$$\ell''(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} y_i^r$$

so that

$$\ell''\left(\frac{1}{n} \sum y_i^r\right) = -\frac{n^3}{\left(\sum y_i^r\right)^2} < 0.$$

By the second derivative test we conclude,

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} Y_i^r.$$

(9.75) (a) Suppose that $Y_1, \ldots, Y_n$ are iid uniform$[0, 2\theta+1]$ so that each Y_i has density function

$$f(y|\theta) = \frac{1}{2\theta + 1}, \quad 0 \le y \le 2\theta + 1.$$

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = (2\theta+1)^{-n}$$

provided that $0 \le y_i \le 2\theta + 1$ for each $i = 1, \ldots, n$. In other words,

$$L(\theta) = \begin{cases} (2\theta+1)^{-n}, & 0 \le \max\{y_1, \ldots, y_n\} \le 2\theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ is obtained by maximizing $L(\theta)$. Since $(2\theta + 1)^{-n}$ is strictly decreasing in θ provided that $0 \le \max\{y_1, \ldots, y_n\} \le 2\theta + 1$, we see that the maximum value is obtained when θ is chosen as small as possible subject to the constraint $\max\{y_1, \ldots, y_n\} \le 2\theta + 1$. Thus, the maximum likelihood estimator is

$$\hat{\theta}_{\text{MLE}} = \frac{\max\{Y_1, \ldots, Y_n\} - 1}{2}.$$

(9.76) (a) Suppose that Y_1, Y_2, Y_3 are iid Gamma$(2, \theta)$ random variables so that each has density

$$f(y|\theta) = \frac{1}{\theta^2} y e^{-y/\theta}, \quad y > 0$$

Therefore, the likelihood function is

$$L(\theta) = \prod_{i=1}^{3} f(y_i|\theta) = \prod_{i=1}^{3} \frac{1}{\theta^2} y_i e^{-y_i/\theta} = \theta^{-6} \cdot \prod_{i=1}^{3} y_i \cdot \exp\left(-\frac{1}{\theta}\sum_{i=1}^{3} y_i\right).$$

The maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ is obtained by maximizing $L(\theta)$, or equivalently, by maximizing the log-likelihood function $\ell(\theta)$ given by

$$\ell(\theta) = -6\log\theta + \sum_{i=1}^{3} \log y_i - \frac{1}{\theta}\sum_{i=1}^{3} y_i.$$

Since

$$\ell'(\theta) = -\frac{6}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{3} y_i$$

we find $\ell'(\theta) = 0$ when

$$-\frac{6}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{3} y_i = 0 \quad \text{or} \quad \theta = \frac{1}{6}\sum_{i=1}^{3} y_i = \frac{\overline{y}}{2}.$$

Finally

$$\ell''(\theta) = \frac{6}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{3} y_i$$

so that

$$\ell''\left(\frac{1}{6}\sum y_i\right) = -\frac{6^3}{\left(\sum y_i\right)^2} < 0.$$

By the second derivative test we conclude,

$$\hat{\theta}_{\text{MLE}} = \frac{1}{6}\sum_{i=1}^{3} Y_i = \frac{\overline{Y}}{2}.$$

Based on the observed data, we find that the maximum likelihood estimate of θ is

$$\hat{\theta}_{\text{MLE}} = \frac{120 + 130 + 128}{6} = 63.$$

(b) Since each Y_1, Y_2, Y_3 are iid Gamma$(2, \theta)$, and since

$$\hat{\theta}_{\text{MLE}} = \frac{1}{6}\sum_{i=1}^{3} Y_i = \frac{\overline{Y}}{2}$$

we conclude

$$E(\hat{\theta}_{\text{MLE}}) = \frac{1}{6}\sum_{i=1}^{3} E(Y_i) = \frac{3\cdot 2\theta}{6} = \theta$$

and

$$\text{Var}(\hat{\theta}_{\text{MLE}}) = \frac{1}{6^2}\sum_{i=1}^{3} \text{Var}(Y_i) = \frac{3\cdot 2\theta^2}{6^2} = \frac{\theta^2}{6}.$$

(c) If $\theta = 130$, then an approximate bound for the error of estimation is given by

$$2\sqrt{\mathrm{Var}(\hat{\theta}_{\mathrm{MLE}})} = 2\sqrt{\frac{\theta^2}{6}} = 2\sqrt{\frac{130^2}{6}} \approx 106.14.$$

(d) Since the variance of Y is $\mathrm{Var}(Y) = 2\theta^2$, we conclude that the MLE of $\mathrm{Var}(Y)$ is

$$\hat{\mathrm{Var}(Y)}_{\mathrm{MLE}} = 2\hat{\theta}^2_{\mathrm{MLE}} = 2(63)^2 = 7938.$$

(9.77) **(a)** Suppose that $Y_1, \ldots, Y_n$ are a random sample from the density function

$$f(y|\theta) = \frac{1}{\Gamma(\alpha)\theta^\alpha} y^{\alpha-1} e^{-y/\theta}, \quad y > 0$$

where $\alpha > 0$ is known. Therefore, the likelihood function is

$$L(\theta) = \prod_{i=1}^n f(y_i|\theta) = \prod_{i=1}^n \frac{1}{\Gamma(\alpha)\theta^\alpha} y_i^{\alpha-1} e^{-y_i/\theta} = \theta^{-n\alpha} \cdot \Gamma(\alpha)^{-n} \cdot \left(\prod_{i=1}^n y_i\right)^{\alpha-1} \cdot \exp\left(-\frac{1}{\theta}\sum_{i=1}^n y_i\right).$$

The maximum likelihood estimator $\hat{\theta}_{\mathrm{MLE}}$ is obtained by maximizing $L(\theta)$, or equivalently, by maximizing the log-likelihood function $\ell(\theta)$ given by

$$\ell(\theta) = -n\alpha \log\theta - n\log\Gamma(\alpha) + (\alpha - 1)\sum_{i=1}^n \log y_i - \frac{1}{\theta}\sum_{i=1}^n y_i.$$

Since

$$\ell'(\theta) = -\frac{n\alpha}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n y_i$$

we find $\ell'(\theta) = 0$ when

$$-\frac{n\alpha}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n y_i = 0 \quad \text{or} \quad \theta = \frac{1}{n\alpha}\sum_{i=1}^n y_i = \frac{\overline{y}}{\alpha}.$$

Finally

$$\ell''(\theta) = \frac{n\alpha}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^n y_i$$

so that

$$\ell''\left(\frac{1}{n\alpha}\sum y_i\right) = -\frac{n^3\alpha^3}{\left(\sum y_i\right)^2} < 0$$

since $\alpha > 0$. By the second derivative test we conclude,

$$\hat{\theta}_{\mathrm{MLE}} = \frac{1}{n\alpha}\sum_{i=1}^n Y_i = \frac{\overline{Y}}{\alpha}.$$