Stat 252.01 Winter 2006
Assignment #6 Solutions

Exercises from Text

(8.39) (a) If $Y_1, \ldots, Y_n$ are iid uniform$[0, \theta]$, then for $0 \le t \le \theta$,

$$P(Y_{(n)} \le t) = P(Y_1 \le t, \ldots, Y_n \le t) = [P(Y_1 \le t)]^n = \left(\frac{t}{\theta}\right)^n.$$

Hence, if $U = Y_{(n)}/\theta$, then

$$P(U \le u) = P(Y_{(n)} \le u\theta) = \left(\frac{u\theta}{\theta}\right)^n = u^n, \quad 0 \le u \le 1.$$

That is,

$$F_U(u) = \begin{cases} 0, & u < 0 \\ u^n, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

(b) A 95% lower confidence bound for θ is therefore found by finding a such that $P(U > a) = 0.05$ for then we will have

$$P(U > a) = P\left(\frac{Y_{(n)}}{\theta} > a\right) = P\left(\theta < \frac{Y_{(n)}}{a}\right) = 0.05.$$

Solving

$$0.05 = P(U > a) = \int_a^1 f_u(u)\, du = \int_a^1 nu^{n-1}\, du = 1 - a^n$$

for a gives $a = (0.95)^{1/n}$ so that

$$P\left(\theta < \frac{Y_{(n)}}{(0.95)^{1/n}}\right) = 0.05 \quad \text{or, equivalently,} \quad P\left(\theta \ge \frac{Y_{(n)}}{(0.95)^{1/n}}\right) = 0.95.$$

(8.40) (a) By definition, if $0 < y < \theta$, then

$$F_Y(y) = \int_{-\infty}^y f(u)\, du = \int_0^y \frac{2(\theta - u)}{\theta^2}\, du = \left.\frac{(2\theta u - u^2)}{\theta^2}\right|_0^y = \frac{(2\theta y - y^2)}{\theta^2} = \frac{2y}{\theta} - \frac{y^2}{\theta^2}.$$

That is,

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ \frac{2y}{\theta} - \frac{y^2}{\theta^2}, & 0 < y < 1, \\ 1, & y \ge \theta. \end{cases}$$

(b) If $U = Y/\theta$, then for $0 < u < 1$,

$$F_U(u) = P(U \le u) = P(Y \le u\theta) = \frac{2u\theta}{\theta} - \frac{u^2\theta^2}{\theta^2} = 2u - u^2.$$

Since the distribution of U does not depend on θ, this shows that $U = Y/\theta$ is a pivotal quantity.

(c) A 90% lower confidence limit for θ is therefore found by finding a such that $P(U > a) = 0.10$ for then we will have

$$P(U > a) = P\left(\frac{Y}{\theta} > a\right) = P\left(\theta < \frac{Y}{a}\right) = 0.10.$$

Solving

$$0.10 = P(U > a) = \int_a^1 f_u(u)\, du = \int_a^1 (2 - 2u)\, du = 1 - 2a + a^2$$

for a gives $a = 1 - \sqrt{0.10}$ (use the quadratic formula, and reject the root for which $a > 1$) so that

$$P\left(\theta < \frac{Y}{1 - \sqrt{0.10}}\right) = 0.10 \quad \text{or, equivalently,} \quad P\left(\theta \geq \frac{Y}{1 - \sqrt{0.10}}\right) = 0.90.$$

(8.41) (a) A 90% upper confidence limit for θ is found by finding b such that $P(U < b) = 0.10$ for then we will have

$$P(U < b) = P\left(\frac{Y}{\theta} < b\right) = P\left(\theta > \frac{Y}{b}\right) = 0.10.$$

Solving

$$0.10 = P(U < b) = \int_0^b f_u(u)\, du = \int_0^b (2 - 2u)\, du = 1 - 2b + b^2$$

for b gives $b = 1 - \sqrt{0.90}$ (use the quadratic formula, and reject the root for which $b > 1$) so that

$$P\left(\theta > \frac{Y}{1 - \sqrt{0.90}}\right) = 0.10 \quad \text{or, equivalently,} \quad P\left(\theta \leq \frac{Y}{1 - \sqrt{0.90}}\right) = 0.90.$$

(b) We know from **(8.40) (c)** that

$$P\left(\theta < \frac{Y}{1 - \sqrt{0.10}}\right) = 0.10$$

and we know from **(8.41) (a)** that

$$P\left(\theta > \frac{Y}{1 - \sqrt{0.90}}\right) = 0.10.$$

Therefore,

$$P\left(\frac{Y}{1 - \sqrt{0.90}} < \theta < \frac{Y}{1 - \sqrt{0.10}}\right) = 0.80$$

so that

$$\left(\frac{Y}{1 - \sqrt{0.90}}, \frac{Y}{1 - \sqrt{0.10}}\right)$$

is an 80% confidence interval for θ.

(8.6) Recall that a Poisson(λ) random variable has mean λ and variance λ. This was also done in Stat 251.

(a) Since λ is the mean of a Poisson(λ) random variable, then a natural unbiased estimator for λ is

$$\hat{\lambda} = \overline{Y}.$$

(As you saw in problem **(8.4)**, there is NO unique unbiased estimator, so many other answers are possible.) It is a simple matter to compute that

$$\mathbb{E}(\hat{\lambda}) = \mathbb{E}(\overline{Y}) = \lambda \quad \text{and} \quad \mathrm{Var}(\hat{\lambda}) = \frac{\lambda}{n}.$$

We will need these in **(c)**.

(b) If $C = 3Y + Y^2$, then

$$\mathbb{E}(C) = \mathbb{E}(3Y) + \mathbb{E}(Y^2) = 3\mathbb{E}(Y) + [\mathrm{Var}(Y) + \mathbb{E}(Y)^2] = 3\lambda + [\lambda + \lambda^2] = 4\lambda + \lambda^2.$$

(c) This part is a little tricky. There is NO algorithm to solve it; instead you must THINK. Since $\mathbb{E}(C)$ depends on the *parameter* λ, we do not know its actual value. Therefore, we can *estimate* it. Suppose that $\theta = \mathbb{E}(C)$. Then, a *natural* estimator of $\theta = 4\lambda + \lambda^2$ is

$$\hat{\theta} = 4\hat{\lambda} + \hat{\lambda}^2,$$

where $\hat{\lambda} = \overline{Y}$ as in **(a)**. However, if we compute $\mathbb{E}(\hat{\lambda})$ we find

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(4\hat{\lambda}) + \mathbb{E}(\hat{\lambda}^2) = 4\mathbb{E}(\hat{\lambda}) + [\mathrm{Var}(\hat{\lambda}) + \mathbb{E}(\hat{\lambda})^2] = 4\lambda + \frac{\lambda}{n} + \lambda^2.$$

This does not equal θ, so that $\hat{\theta}$ is NOT unbiased. However, a little thought shows that if we *define*

$$\tilde{\theta} := 4\hat{\lambda} + \hat{\lambda}^2 - \frac{\hat{\lambda}}{n} = 4\overline{Y} + \overline{Y}^2 - \frac{\overline{Y}}{n}$$

then, $\mathbb{E}(\tilde{\theta}) = 4\hat{\lambda} + \hat{\lambda}^2$ so that $\tilde{\theta}$ IS an unbiased estimator of $\theta = \mathbb{E}(C)$.

(8.8) If Y is a uniform $(\theta, \theta + 1)$ random variable, then its density is

$$f(y) = \begin{cases} 1, & \theta \le y \le \theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is a simple matter to compute

$$\mathbb{E}(Y) = \frac{2\theta + 1}{2} \quad \text{and} \quad \mathrm{Var}\, Y = \frac{1}{12}.$$

(a) Hence,

$$\mathbb{E}(\overline{Y}) = \mathbb{E}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_n)}{n} = \frac{\frac{2\theta+1}{2} + \cdots + \frac{2\theta+1}{2}}{n} = \frac{2n\theta + n}{2n} = \theta + \frac{1}{2}.$$

We now find

$$B(\overline{Y}) = \mathbb{E}(\overline{Y}) - \theta = \left(\theta + \frac{1}{2}\right) - \theta = \frac{1}{2}.$$

(b) A little thought shows that our calculation in **(a)** iummediately suggests a natural unbiased estimator of θ, namely

$$\hat{\theta} = \overline{Y} - \frac{1}{2}.$$

(c) We first compute that

$$\text{Var}(\overline{Y}) = \text{Var}\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = \frac{\text{Var}(Y_1) + \cdots + \text{Var}(Y_n)}{n^2} = \frac{1/12 + \cdots + 1/12}{n^2} = \frac{1}{12n}.$$

As on page 367,

$$MSE(\overline{Y}) = \text{Var}(\overline{Y}) + (B(\overline{Y}))^2$$

so that

$$MSE(\overline{Y}) = \frac{1}{12n} + \left(\frac{1}{2}\right)^2 = \frac{3n+1}{12n}.$$

(8.9) (a) Let $\theta = \text{Var}(Y)$, and $\hat{\theta} = n(Y/n)(1 - Y/n)$. To prove $\hat{\theta}$ is unbiased, we must show that $\mathbb{E}(\hat{\theta}) \neq \theta$. Since

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(n(Y/n)(1 - Y/n)) = \mathbb{E}(Y) - \frac{1}{n}\mathbb{E}(Y^2),$$

and since Y is Binomial(n, p) so that $\mathbb{E}(Y) = np$, $\mathbb{E}(Y^2) = \text{Var}(Y) + [\mathbb{E}(Y)]^2 = np(1-p) + n^2p^2$, we conclude that

$$\mathbb{E}(\hat{\theta}) = np - \frac{np(1-p) + n^2p^2}{n} = (n-1)p(1-p).$$

(b) As an unbiased estimator, use

$$\frac{n}{n-1}\hat{\theta} = n\left(\frac{Y}{n-1}\right)\left(1 - \frac{Y}{n}\right).$$

(8.34) Let $\theta = V(Y)$. If Y is a geometric random variable, then

$$\mathbb{E}(Y^2) = V(Y) + [\mathbb{E}(Y)]^2 = \frac{2}{p^2} - \frac{1}{p}.$$

Now a little thought shows that

$$\mathbb{E}\left(\frac{Y^2}{2} - \frac{Y}{2}\right) = \frac{1}{p^2} - \frac{1}{2p} - \frac{1}{2p} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \theta.$$

Thus, choose

$$\hat{V}(Y) = \hat{\theta} = \frac{Y^2 - Y}{2}.$$

If Y is used to estimate $1/p$, then a two standard error bound on the error of estimation is given by

$$2\sqrt{\hat{V}(Y)} = 2\sqrt{\hat{\theta}} = 2\sqrt{\frac{Y^2 - Y}{2}}.$$

(8.58) (a) As noted in class (see Example 8.9), we need to solve the equation

$$1.96\sqrt{\frac{p(1-p)}{n}} = 0.05$$

for n when $p = 0.9$. We find $n = 138.2976$, so that we take a sample size of $n = 139$. (Note that we cannot have a fractional sample size and that we need to round *up* to 139 because if we round down, then the variance will be *more* than 0.05.)

(b) If no information about p is known, then using $p = 0.5$ is the most conservative estimate. In this case, we solve

$$1.96\sqrt{\frac{p(1-p)}{n}} = 0.05$$

for n when $p = 0.5$. This gives a sample size of $n = 385$.

(8.4) (a) Recall that if Y has the exponential density as given in the problem, then $\mathbb{E}(Y) = \theta$. This was done in Stat 251. In order to decide which estimators are unbiased, we simply compute $\mathbb{E}(\hat{\theta}_i)$ for each i. Four of these are easy:

$$\begin{aligned}
\mathbb{E}(\hat{\theta}_1) &= \mathbb{E}(Y_1) = \theta; \\
\mathbb{E}(\hat{\theta}_2) &= \mathbb{E}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\mathbb{E}(Y_1) + \mathbb{E}(Y_2)}{2} = \frac{\theta + \theta}{2} = \theta; \\
\mathbb{E}(\hat{\theta}_3) &= \mathbb{E}\left(\frac{Y_1 + 2Y_2}{3}\right) = \frac{\mathbb{E}(Y_1) + 2\mathbb{E}(Y_2)}{3} = \frac{\theta + 2\theta}{3} = \theta; \\
\mathbb{E}(\hat{\theta}_5) &= \mathbb{E}(\overline{Y}) = \mathbb{E}\left(\frac{Y_1 + Y_2 + Y_3}{3}\right) = \frac{\mathbb{E}(Y_1) + \mathbb{E}(Y_2) + \mathbb{E}(Y_3)}{3} = \frac{\theta + \theta + \theta}{3} = \theta.
\end{aligned}$$

In order to compute $\mathbb{E}(\hat{\theta}_4) = \mathbb{E}(\min(Y_1, Y_2, Y_3))$ we need to do a bit of work.

$$\begin{aligned}
P(\min(Y_1, Y_2, Y_3) > t) = P(Y_1 > t, Y_2 > t, Y_3 > t) &= P(Y_1 > t) \cdot P(Y_2 > t) \cdot P(Y_3 > t) \\
&= [P(Y_1 > t)]^3 \\
&= e^{-3t/\theta}.
\end{aligned}$$

Thus, $f(t) = (3/\theta)e^{-3t/\theta}$ which, as you will notice, is the density of an Exponential($\theta/3$) random variable. (WHY?) Thus,

$$\mathbb{E}(\hat{\theta}_4) = \mathbb{E}(\min(Y_1, Y_2, Y_3)) = \frac{\theta}{3}.$$

Hence, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, and $\hat{\theta}_5$ are unbiased, while $\hat{\theta}_4$ is biased.

(b) To decide which has the smallest variance, we simply compute. Recall that an Exponential(θ) random variable has variance θ^2. Thus,

$$\begin{aligned}
\text{Var}(\hat{\theta}_1) &= \text{Var}(Y_1) = \theta^2; \\
\text{Var}(\hat{\theta}_2) &= \text{Var}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\text{Var}(Y_1) + \text{Var}(Y_2)}{4} = \frac{\theta^2 + \theta^2}{4} = \frac{\theta^2}{2}; \\
\text{Var}(\hat{\theta}_3) &= \text{Var}\left(\frac{Y_1 + 2Y_2}{3}\right) = \frac{\text{Var}(Y_1) + 4\,\text{Var}(Y_2)}{9} = \frac{\theta^2 + 4\theta^2}{9} = \frac{5\theta^2}{9}; \\
\text{Var}(\hat{\theta}_5) &= \text{Var}(\overline{Y}) = \text{Var}\left(\frac{Y_1 + Y_2 + Y_3}{3}\right) = \frac{\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3)}{9} = \frac{\theta^2 + \theta^2 + \theta^2}{9} = \frac{\theta^2}{3}.
\end{aligned}$$

Thus, $\hat{\theta}_5$ has the smallest variance. In fact, we will show later that it is the *minimum variance unbiased estimator*. That is, *no* other unbiased estimator of the mean will have smaller variance than $\overline{Y}$.

(9.1) Using the results of Exercise 8.4, we find

$$\operatorname{Var}(\hat{\theta}_1) = \theta^2, \quad \operatorname{Var}(\hat{\theta}_2) = \frac{\theta^2}{2}, \quad \operatorname{Var}(\hat{\theta}_3) = \frac{5\theta^2}{9}, \quad \operatorname{Var}(\hat{\theta}_5) = \frac{\theta^2}{3}.$$

Thus,

$$\operatorname{eff}(\hat{\theta}_1, \hat{\theta}_5) = \frac{\operatorname{Var}(\hat{\theta}_5)}{\operatorname{Var}(\hat{\theta}_1)} = \frac{1}{3}, \quad \operatorname{eff}(\hat{\theta}_2, \hat{\theta}_5) = \frac{\operatorname{Var}(\hat{\theta}_5)}{\operatorname{Var}(\hat{\theta}_2)} = \frac{2}{3}, \quad \operatorname{eff}(\hat{\theta}_3, \hat{\theta}_5) = \frac{\operatorname{Var}(\hat{\theta}_5)}{\operatorname{Var}(\hat{\theta}_3)} = \frac{3}{5}.$$

(9.4) In Example 9.1, it is shown that

$$\operatorname{Var}(\hat{\theta}_2) = \frac{\theta^2}{n(n+2)},$$

and we have as a simple extension of Problem #1 on Assignment #2 that

$$\operatorname{Var}(\hat{\theta}_1) = (n+1)^2 \operatorname{Var}(Y_{(1)}) = (n+1)^2 \left[\frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} \right] = \frac{n\theta^2}{n+2}.$$

Thus we conclude,

$$\operatorname{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\operatorname{Var}(\hat{\theta}_2)}{\operatorname{Var}(\hat{\theta}_1)} = \frac{1}{n^2}.$$

Notice that this result implies that

$$\operatorname{Var}(\hat{\theta}_1) = n^2 \operatorname{Var}(\hat{\theta}_2).$$

As n increases, the variance of $\hat{\theta}_1$ increases very quickly relative to the variance of $\hat{\theta}_2$. In other words, the larger n, the bigger the variance of $\hat{\theta}_1$ relative to variance $\hat{\theta}_2$. Thus, $\hat{\theta}_2$ is a markedly superior (unbiased) estimator.

(9.7) If $\operatorname{MSE}(\hat{\theta}_1) = \theta^2$, then $\operatorname{Var}(\hat{\theta}_1) = \operatorname{MSE}(\hat{\theta}_1) = \theta^2$ since $\hat{\theta}_1$ is unbiased. If $\hat{\theta}_2 = \overline{Y}$, then since the Y_i are exponential, we conclude $\mathbb{E}(\overline{Y}) = \theta$ and $\operatorname{Var}(\overline{Y}) = \theta^2/n$. Thus,

$$\operatorname{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\operatorname{Var}(\hat{\theta}_2)}{\operatorname{Var}(\hat{\theta}_1)} = \frac{1}{n}.$$

Extra Exercise

Suppose that

$$f(y|\theta) = \frac{e^{(y-\theta)}}{\left[1 + e^{(y-\theta)}\right]^2},$$

where $-\infty < y < \infty$, and $-\infty < \theta < \infty$. Let $U = Y - \theta$ so that

$$P(U \le u) = P(Y \le u + \theta) = \int_{-\infty}^{u+\theta} \frac{e^{(y-\theta)}}{\left[1 + e^{(y-\theta)}\right]^2} \, dy = \left. \frac{e^{(y-\theta)}}{1 + e^{(y-\theta)}} \right|_{-\infty}^{u+\theta} = \frac{e^u}{1 + e^u}.$$

We also calculate that for $-\infty < u < \infty$,

$$f_U(u) = \frac{e^u}{[1 + e^u]^2}.$$

Therefore, U is a pivotal quantity. Hence, we must find a and b so that

$$\int_{-\infty}^{a} f_U(u)\,du = \alpha_1 \quad \text{and} \quad \int_{b}^{\infty} f_U(u)\,du = \alpha_2.$$

Now,

$$\alpha_1 = \int_{-\infty}^{a} f_U(u)\,du = \int_{-\infty}^{a} \frac{e^u}{[1+e^u]^2}\,du = \left.\frac{e^u}{1+e^u}\right|_{-\infty}^{a} = \frac{e^a}{1+e^a}$$

so that

$$a = \log\left(\frac{\alpha_1}{1-\alpha_1}\right).$$

Furthermore,

$$\alpha_2 = \int_{b}^{\infty} f_U(u)\,du = \int_{b}^{\infty} \frac{e^u}{[1+e^u]^2}\,du = \left.\frac{e^u}{1+e^u}\right|_{b}^{\infty} = 1 - \frac{e^b}{1+e^b}$$

so that

$$b = \log\left(\frac{1-\alpha_2}{\alpha_2}\right).$$

This tells us that

$$1-(\alpha_1+\alpha_2) = P(a \le U \le b) = P\left(\log\left(\frac{\alpha_1}{1-\alpha_1}\right) \le U \le \log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right)$$

or, in other words,

$$\begin{aligned} 1-(\alpha_1+\alpha_2) &= P\left(\log\left(\frac{\alpha_1}{1-\alpha_1}\right) \le Y-\theta \le \log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) \\ &= P\left(Y - \log\left(\frac{1-\alpha_2}{\alpha_2}\right) \le \theta \le Y - \log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right). \end{aligned}$$