Stat 252.01 Winter 2006
Assignment #3 Solutions

(6.64) If $Y_1, Y_2, \ldots, Y_n$ are all independent and identically distributed beta$(2,2)$ random variables, then each has density function

$$f_Y(y) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} y(1-y) = 6y(1-y), \;\; 0 < y < 1.$$

(a) If $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$, then

$$P(Y_{(n)} \leq t) = P(Y_1 \leq t, \ldots, Y_n \leq t) = P(Y_1 \leq t) \cdots P(Y_n \leq t) = [P(Y_1 \leq t)]^n$$

since the Y_i are independent (the second equality) and identically distributed (the third equality). Now, for any $0 < t < 1$,

$$P(Y_1 \leq t) = \int_0^t f_Y(y)\, dy = \int_0^t 6y(1-y)\, dt = \int_0^t 6y\, dy - \int_0^t 6y^2\, dy = 3t^2 - 2t^3 = t^2(3-2t)$$

so that the distribution function of $Y_{(n)}$ is

$$F(t) = [P(Y_{(n)} \leq t)]^n = t^{2n}(3-2t)^n, \;\; 0 < t < 1.$$

(Of course, $F(t) = 0$ for $t \leq 0$, and $F(t) = 1$ for $t \geq 1$.

(b) The density function of $Y_{(n)}$ is therefore

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}t^{2n}(3-2t)^n = 2nt^{2n-1}(3-2t)^n - 2nt^{2n}(3-2t)^{n-1} = 6nt^{2n-1}(3-2t)^{n-1}(1-t)$$

for $0 < t < 1$ and 0 otherwise.

(c) For $n = 2$, the expected value $E(Y_{(2)})$ is

$$E(Y_{(2)}) = \int_0^1 t\, f(t)\, dt = \int_0^1 12t^4(3-2t)(1-t)\, dt = \int_0^1 (36t^4 - 60t^5 + 24t^6)\, dt = \frac{36}{5} - \frac{60}{6} + \frac{24}{7} = \frac{22}{35}.$$

(6.65) If $Y_1, Y_2, \ldots, Y_n$ are all independent and identically distributed exponential(β) random variables, then each has density function

$$f_Y(y) = \frac{1}{\beta}e^{-y/\beta}, \;\; 0 < y < \infty.$$

(a) If $Y_{(1)} = \min\{Y_1, \ldots, Y_n\}$, then

$$P(Y_{(1)} > t) = P(Y_1 > t, \ldots, Y_n > t) = P(Y_1 > t) \cdots P(Y_n > t) = [P(Y_1 > t)]^n$$

since the Y_i are independent (the second equality) and identically distributed (the third equality). Now, for any $0 < t < \infty$,

$$P(Y_1 > t) = \int_t^\infty f_Y(y)\, dy = \frac{1}{\beta}\int_t^\infty e^{-y/\beta}\, dy = e^{-t/\beta}$$

so that

$$P(Y_{(1)} > t) = e^{-tn/\beta}.$$

Thus, the distribution function of $Y_{(n)}$ is

$$F(t) = P(Y_{(1)} \leq t) = 1 - P(Y_{(1)} > t) = 1 - e^{-tn/\beta} = 1 - e^{-t/(\beta/n)}$$

which is the distribution function of an exponential random variable with mean β/n.

(b) If $n = 5$, $\beta = 2$, then the distribution function of $Y_{(1)}$ is

$$F(t) = 1 - e^{-5t/2}, \quad 0 < t < \infty$$

so that the corresponding density function is

$$f(t) = \frac{5}{2}e^{-5t/2}, \quad 0 < t < \infty.$$

Hence,

$$P(Y_{(1)} \le 3.6) = \frac{5}{2}\int_0^{3.6} e^{-5t/2}\,dt = -e^{-5t/2}\Big|_0^{3.6} = 1 - e^{-9}.$$

2. (a) Since $Y_1 \sim \mathcal{U}(0, \theta)$, we have

$$f_Y(y) = \begin{cases} 1/\theta, & 0 \le y \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If $f(t)$ denotes the density of $\hat{\theta} = \min(Y_1, \ldots, Y_n)$, then $f(t) = F'(t)$, where

$$F(t) = P(\hat{\theta} \le t) = 1 - P(\hat{\theta} > t) = 1 - P(Y_1 > t, \ldots, Y_n > t) = 1 - [P(Y_1 > t)]^n.$$

Note that in the last step we have used the fact that Y_i are iid. Next, for $0 \le t \le \theta$, we compute

$$P(Y_1 > t) = \int_t^\infty f_Y(y)\,dy = \int_t^\theta \frac{1}{\theta}\,dy = \frac{\theta - t}{\theta}.$$

Thus, we conclude

$$f(t) = \frac{d}{dt}\left(1 - \left[\frac{\theta - t}{\theta}\right]^n\right) = n\theta^{-n}(\theta - t)^{n-1}, \quad 0 \le t \le \theta.$$

(c) By definition,

$$E(\hat{\theta}) = \int_{-\infty}^{\infty} t f(t)\,dt = \int_0^\theta n\theta^{-n} t(\theta - t)^{n-1}\,dt.$$

This last integral is solved with a simple substitution. Let $u = \theta - t$ so that $du = -dt$. Thus,

$$\begin{aligned}\int_0^\theta n\theta^{-n} t(\theta - t)^{n-1}\,dt = -n\theta^{-n}\int_\theta^0 (\theta - u)u^{n-1}\,du &= n\theta^{-n}\int_0^\theta \theta u^{n-1} - u^n\,du \\ &= n\theta^{-n}\left(\theta \cdot \frac{\theta^n}{n} - \frac{\theta^{n+1}}{n+1}\right) \\ &= \frac{\theta}{n+1}.\end{aligned}$$

(d) From **(c)**, we clearly see that $\hat{\theta}$ is NOT an unbiased estimator of θ. However,

$$\tilde{\theta} = (n+1)\min(Y_1, \ldots, Y_n)$$

IS an unbiased estimator of θ. (You should check that $2\overline{Y}$ is also an unbiased estimator of θ. Why?)