Statistics 252 Winter 2005 Midterm #2 – Solutions

1. (a) By definition, the significance level α is the probability of a Type I error; that is, the probability under H_0 that H_0 is rejected. Hence, since $\overline{Y} \sim \mathcal{N}(\mu, 9/n)$,

$$0.05 = P_{H_0}(\text{reject } H_0) = P(\overline{Y} > c|\mu = 0) = P\left(\frac{\overline{Y} - 0}{3/\sqrt{n}} > \frac{c - 0}{3/\sqrt{n}}\right) = P(Z > c\sqrt{n}\,/\,3),$$

where $Z \sim \mathcal{N}(0, 1)$. From Table 4, we find that $P(Z > 1.65) = 0.05$. We, therefore, must have

$$\frac{c\sqrt{n}}{3} = 1.96 \quad \text{or} \quad c = \frac{4.95}{\sqrt{n}}.$$

1. (b) By definition, the power of an hypothesis test is the probability under H_A that H_0 is rejected. Hence, when $\mu = 1$, $n = 36$, we find $c = 0.825$, so that

$$\text{power} = P(\overline{Y} > 0.825|\mu = 1) = P\left(\frac{\overline{Y} - 1}{3/\sqrt{36}} > \frac{0.825 - 1}{3/\sqrt{36}}\right) = P(Z > -0.36) = 1 - 0.3594 = 0.6406$$

where $Z \sim \mathcal{N}(0, 1)$. (The last step follows from Table 4.)

1. (c) As in **(a)** and **(b)**,

$$\text{power} = P\left(\overline{Y} > \frac{4.95}{\sqrt{n}}|\mu = 1\right) = P\left(Z > \frac{4.95/\sqrt{n} - 1}{3/\sqrt{n}}\right) = P(Z > 1.65 - \sqrt{n}/3).$$

Hence, as n increases ($\to \infty$), $1.65 - \sqrt{n}/3$ decreases monotonically ($\to -\infty$), so that the the power increases monotonically ($\to 1$). In particular, if $m > n$, then

$$P(Z > 1.65 - \sqrt{n}/3\,) < P(Z > 1.65 - \sqrt{m}/3\,).$$

This indeed makes sense intuitively. As the sample size increases, it becomes easier to detect that $\mu = 1$ is false.

2. Draw a picture! From the scenario presented, we know that John rejects H_0 iff $p \leq 0.01$, and that George rejects H_0 iff $p \leq 0.05$. Since Ringo's p-value is smaller than 0.03, we can conclude immediately that George will reject the null hypothesis. However, John cannot make a decision. We are only told that Ringo's p-value is smaller than 0.03. We do not know, therefore, how it compares to John's desired significance level of $\alpha = 0.01$. (It could be the case that $0.01 < p < 0.03$ or it could be the case that $p < 0.01 < 0.03$. These yield different conclusions for John.)

3. Confidence intervals and hypothesis tests are dual in the sense that for a fixed value of α, a level $1 - \alpha$ confidence interval for a parameter θ will contain a specified value θ if and only if a significance level hypothesis test with $H_0 : \theta = \theta_0$ will fail to reject H_0. This means that confidence intervals and hypothesis tests provide two distinct frameworks for answering the same questions.

4. (a) Recall that the generalized likelihood ratio test for the simple null hypothesis $H_0 : \theta = \theta_0$ against the composite alternative hypothesis $H_A : \theta \neq \theta_0$ has rejection region $\{\Lambda \leq c\}$ where Λ is the generalized likelihood ratio

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{\text{MLE}})}$$

where $L(\theta)$ is the likelihood function. In this instance,

$$L(\theta) = \theta^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp \left(-\theta \sum_{i=1}^{n} y_i \right)$$

so that

$$\Lambda = \frac{\theta_0^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp \left(-\theta_0 \sum_{i=1}^{n} y_i \right)}{\hat{\theta}_{\text{MLE}}^{2n} \left(\prod_{i=1}^{n} y_i \right) \exp \left(-\hat{\theta}_{\text{MLE}} \sum_{i=1}^{n} y_i \right)} = \left(\frac{1}{2/\overline{Y}} \right)^{2n} \exp \left(-\sum y_i + 2/\overline{Y} \cdot \sum y_i \right)$$

$$= \left(\frac{\overline{Y}}{2} \right)^{2n} \exp \left(2n - n\overline{Y} \right).$$

4. (b) We saw in class that $-2 \log \Lambda \sim \chi_1^2$ (approximately). This means that the generalized likelihood ratio test rejection region is $\{\Lambda \le c\} = \{-2 \log \Lambda \ge K\}$ where $K = -2 \log c$ is (yet another) constant. (In fact, $K = \chi_{1,\alpha}^2$.) As we found above,

$$\Lambda = \left(\frac{\overline{Y}}{2} \right)^{2n} \exp \left(2n - n\overline{Y} \right)$$

so that

$$-2 \log \Lambda = -4n \log \overline{Y} + 4n \log 2 - 4n + 2n\overline{Y}.$$

Hence, to conduct the GLRT, we need to compare the observed value of $-2 \log \Lambda$ with the appropriate chi-squared critical value which is $\chi_{1,0.05}^2 = 3.84146$. Since

$$-4 \cdot 5 \cdot \log 1 + 4 \cdot 5 \cdot \log 2 - 4 \cdot 5 + 2 \cdot 5 \cdot 1 \approx 3.8629$$

is the observed value of $-2 \log \Lambda$, we reject H_0 at significance level 0.05 (but just barely).

5. As given in the hint,

$$\mathbb{E}[(Y - \hat{Y})^2] = [\mathbb{E}(Y) - \mathbb{E}(\hat{Y})]^2 + \text{Var}(Y - \hat{Y}) = [\mathbb{E}(Y) - \mathbb{E}(\hat{Y})]^2 + \text{Var}(Y) + \text{Var}(\hat{Y}) - 2\,\text{Cov}(Y, \hat{Y}).$$

Now, using the notation given in the problem, and simple properties of expectations and covariances, we find that

- $\mathbb{E}(Y) = \mu_y$;
- $\mathbb{E}(\hat{Y}) = \beta_0 + \beta_1 \mu_x$;
- $\text{Var}(Y) = \sigma_y^2$;
- $\text{Var}(\hat{Y}) = \beta_1^2 \sigma_x^2$; and
- $\text{Cov}(Y, \hat{Y}) = \text{Cov}(Y, \beta_0 + \beta_1 X) = \text{Cov}(Y, \beta_1 X) = \beta_1 \,\text{Cov}(Y, X) = \beta_1 \sigma_{xy}$.

Hence, by combining the above,

$$\mathrm{SSE}(\beta_0, \beta_1) = \mathbb{E}[(Y - \hat{Y})^2] = (\mu_y - \beta_0 - \beta_1\mu_x)^2 + \sigma_y^2 + \beta_1^2\sigma_x^2 - 2\beta_1\sigma_{xy}.$$

Finally, to find the minimizing values of β_0 and β_1, we compute derivatives:

$$\frac{\partial}{\partial\beta_0}\mathrm{SSE}(\beta_0, \beta_1) = -2(\mu_y - \beta_0 - \beta_1\mu_x) \quad \text{and} \quad \frac{\partial}{\partial\beta_1}\mathrm{SSE}(\beta_0, \beta_1) = -2\mu_x(\mu_y - \beta_0 - \beta_1\mu_x) + 2\beta_1\sigma_x^2 - 2\sigma_{xy}.$$

From the first equation,

$$\frac{\partial}{\partial\beta_0}\mathrm{SSE}(\beta_0, \beta_1) = 0$$

implies

$$-2(\mu_y - \beta_0 - \beta_1\mu_x) = 0$$

so that

$$\beta_0 = \mu_y - \beta_1\mu_x.$$

From the second equation,

$$\frac{\partial}{\partial\beta_1}\mathrm{SSE}(\beta_0, \beta_1) = 0$$

implies

$$-2\mu_x(\mu_y - \beta_0 - \beta_1\mu_x) + 2\beta_1\sigma_x^2 - 2\sigma_{xy} = 0$$

so that

$$-2\mu_x(\mu_y - (\mu_y - \beta_1\mu_x) - \beta_1\mu_x) + 2\beta_1\sigma_x^2 - 2\sigma_{xy} = 2\beta_1\sigma_x^2 - 2\sigma_{xy} = 0$$

or, in other words,

$$\beta_1 = \frac{\sigma_{xy}}{\sigma_x^2}.$$