Statistics 252
The Likelihood Ratio Test (March 4, 2005)

Here are some notes about the current Assignment #6. It was clear from the maddening hordes in my office on Friday afternoon that both 10.79 and 10.83 are causing widespread grief.

For any likelihood ratio test, the rejection region is always of the form $\{\Lambda < c\}$ where c is a constant chosen to give the test level α. That is, $c = c(\alpha)$. This *notation* is intended to stress that c is a function of α.

The LR Λ will *always* be a function of the random variables $X_1, \ldots, X_n$ (or, if you prefer, a function of the observations $x_1, \ldots, x_n$). In other words, $\Lambda = \Lambda(X)$.

Thus, our rejection region takes the form $\{\Lambda(X) < c(\alpha)\}$ which, theoretically, can be written as either $\{X > c^*\}$ or $\{X < c^*\}$ where c^* is a new constant which can be written in terms of the old constant c which in turn can be written in terms of α.

During Friday's lecture, you saw this idea. One example involved an exponential random variable, and the LR was

$$\Lambda = \frac{1}{2}e^X.$$

NOTICE THAT Λ IS A FUNCTION OF X. We were also able to EXPLICITLY calculate $c = (2(1-\alpha))^{-1}$. NOTICE THAT c IS A FUNCTION OF α. Thus, as in any LRT, the rejection region is $\{\Lambda < c\}$ which can be written in this case in the more concrete form

$$\{X < -\log(1-\alpha)\}.$$

NOTICE THAT NEW CONSTANTS WERE INTRODUCED WHICH IN TURN WERE FUNCTIONS OF α.

The other example involved a random sample of Poisson random variables. In this case the rejection region was again of the form $\{\Lambda < c\}$. We found Λ explicitly as a function of $\overline{X}$. It was SOMETHING LIKE

$$\{\exp\left(-\left[10 - \overline{X}\left(1 + n\log(10/\overline{X})\right)\right]\right) < c\}.$$

[I am writing this from memory. It might be incorrect.]

The point, however, is that it can be written as

$$\{\overline{X}\left(1 + n\log(10/\overline{X})\right) < c^*\}$$

where $c^* = \log c + 10$ IS JUST ANOTHER CONSTANT.

IMPORTANT

It is not EASY to solve for $\overline{X}$ even though it is theoretically possible. It is also not easy to solve for c (or c^*) in terms of α even though it is also theoretically possible. With a computer, both of these results can be obtained directly enough. But by hand, no way!

However, to stress it yet again, the (G)LRT provides a THEORETICAL FRAMEWORK UPON WHICH ONE CAN ALWAYS CONSTRUCT HYPOTHESIS TESTS.

Now, back to the problems 10.79 and 10.83. In both cases, it is relatively easy to find the likelihood ratio Λ. Thus, in both cases, it is trivial to find the rejection region: that's just $\{\Lambda < c\}$.

HOWEVER, it does take some tedious algebra to actually write things in a more convenient form.

Specifically, for problem 10.79 take a look at Example 10.23 on pages 511–513. You can mimic this example. It is probably wisest not to substitute the maximum likelihood estimator $\hat{\mu}_{\mathrm{MLE}} = \overline{X}$ for μ_a. This is because that would lead to an expression involving $\overline{X}^2$. Yuck. The textbook writes μ_a for the MLE on these pages.

Essentially, you are asked to derive that the LRT for normal distributions is just the "common sense" test we've already studied, namely, reject H_0 if $\{\overline{X} > C\}$ where C is (yet another) constant (not the LRT rejection region constant c from $\{\Lambda < c\}$).

The good thing (or bad thing, perhaps) with the LRT approach is that you can find the explicit form of the constant C in terms of all of the other constants, namely the significance level α (which is written in terms of the LR rejection constant c), the null hypothesis μ_0, the alternative hypothesis μ_a, the sample size n, and the variance of the normal distribution σ^2. This is the centred equation at the top of page 513.

[The textbook writes k for the LRT constant (I wrote c) and k' for the new constant (I wrote C).]

As for problem 10.83: The hint given in **(a)** might be a bit misleading. HOWEVER, you should notice that the density given for Y means that Y has the Gamma$(3, \theta)$ distribution. If you recall from Stat 251, there is a relationship between the Gamma distribution and the χ^2 distribution. In this example (under H_0 so that $\theta = \theta_0$) it amounts to the fact that

$$\frac{2Y}{\theta_0} \sim \chi^2_6.$$

This should explain their hint! [It can be done with a straight-forward change of variables in the integral.]

Also note that the problem asks you to consider FOUR random variables. Remember that the sum of χ^2 random variables is again a χ^2 random variable (but with a different df).