Stat 252.01 Winter 2005
Assignment #10 Solutions

Important Remark: The factorizations of L into $L = g \cdot h$ are *not* unique. Many answers are possible.

Important Remark: Any one-to-one function of a sufficient statistic for θ is also sufficient for θ.

(9.30) If $Y_1, \ldots, Y_n$ are iid $\mathcal{N}(\mu, \sigma^2)$ random variables each with density

$$f(y|\mu, \sigma^2) = \frac{1}{\sqrt{\sigma^2 2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\},$$

then the likelihood function is

$$L(\mu, \sigma^2) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum(y_i - \mu)^2\right\}.$$

(a) If μ is unknown, and σ^2 is known, then with

$$U = \overline{y}, \quad g(U, \mu) = \exp\left\{\frac{1}{2\sigma^2}\left(2\mu n U - \mu^2\right)\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum y_i^2\right\},$$

the Factorization Theorem implies $\overline{Y}$ is sufficient for μ.

(b) If μ is known, and σ^2 is unknown, then with

$$U = \sum(y_i - \mu)^2, \quad g(U, \sigma^2) = (\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}U\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2},$$

the Factorization Theorem implies $\sum(Y_i - \mu)^2$ is sufficient for σ^2.

(c) If both μ and σ^2 are unknown, then with

$$U = (U_1, U_2) = \left(\sum y_i, \sum y_i^2\right),$$

$$g(U, (\mu, \sigma^2)) = g((U_1, U_2), (\mu, \sigma^2)) = (\sigma^2)^{-n/2} \exp\left\{\frac{1}{2\sigma^2}\left(2\mu U_1 + U_2 - \mu^2\right)\right\},$$

$$h(y_1, \ldots, y_n) = (2\pi)^{-n/2},$$

the Factorization Theorem implies $(\sum Y_i, \sum Y_i^2)$ is jointly sufficient for (μ, σ^2).

(9.34) If $Y_1, \ldots, Y_n$ are iid geometric random variables each with density

$$f(y|p) = p(1-p)^y,$$

for $y = 1, 2, 3, \ldots$, then the likelihood function is

$$L(p) = p(1-p)^{\sum y_i}.$$

If

$$U = \overline{y}, \quad g(U,p) = p(1-p)^{nU}, \quad \text{and} \quad h(y_1, \ldots, y_n) = 1,$$

then since $L(p) = g(U,p) \cdot h(y)$, we conclude by the Factorization Theorem that $\overline{Y}$ is sufficient for p.

(9.36) If $Y_1, \ldots, Y_n$ are iid each with density

$$f(y|\alpha, \beta) = \alpha\beta^\alpha y^{-(\alpha+1)}$$

for $y \geq \beta$, then for fixed β the likelihood function is

$$L(\alpha) = \alpha^n \beta^{n\alpha} \left(\prod y_i\right)^{-(\alpha+1)}.$$

If

$$U = \prod y_i, \quad g(U,\alpha) = \alpha^n \beta^{n\alpha} U^{-(\alpha+1)}, \quad \text{and} \quad h(y_1, \ldots, y_n) = 1,$$

then since $L(\alpha) = g(U,\alpha) \cdot h(y)$, we conclude by the Factorization Theorem that $\prod Y_i$ is sufficient for α.

(9.37) If $Y_1, \ldots, Y_n$ are iid each with density from the exponential family

$$f(y|\theta) = a(\theta)b(y)\exp\{c(\theta)d(y)\}, \quad k \leq \theta \leq \ell$$

where k and ℓ do not depend on θ, then the likelihood function is

$$L(\theta) = [a(\theta)]^n \left[\prod b(y_i)\right] \exp\{c(\theta) \sum d(y_i)\}.$$

If

$$U = \sum d(y_i), \quad g(U,\theta) = [a(\theta)]^n \exp\{c(\theta)U\}, \quad \text{and} \quad h(y_1, \ldots, y_n) = \prod b(y_i),$$

then since $L(\theta) = g(U,\theta) \cdot h(y)$, we conclude by the Factorization Theorem that $\sum d(Y_i)$ is sufficient for θ.