Mathematics/Statistics 251 Fall 2015 Midterm #2 – Solutions

1. (a) We find

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} 3(1-x)^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. (b) We find

$$\mathbb{E}(X) = \int_0^1 x \cdot 3(1-x)^2 \,\mathrm{d}x = 3\int_0^1 (x - 2x^2 + x^3)\,\mathrm{d}x = 3\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{1}{4}.$$

1. (c) We find

$$\begin{aligned} \mathbf{P}(-1 < X \le 1/2) = \mathbf{P}(X \le 1/2) - \mathbf{P}(X \le -1) &= F_X(1/2) - F_X(-1) \\ &= \left[1 - (1-1/2)^3\right] - 0 \\ &= 7/8. \end{aligned}$$

1. (d) Let $Y = 1 - X$. If $0 < y < 1$, then

$$\begin{aligned} \mathbf{P}(Y \le y) = \mathbf{P}(1 - X \le y) = P(X \ge 1 - y) = 1 - \mathbf{P}(X \le 1 - y) &= 1 - F_X(1-y) \\ &= 1 - [1 - (1-y)]^3 \\ &= 1 - y^3 \end{aligned}$$

and so the distribution function of Y is

$$F_Y(y) = \begin{cases} 0, & y \le 0, \\ 1 - y^3, & 0 < y < 1, \\ 1, & y \ge 1. \end{cases}$$

2. (a) Let Z denote the random time until either Chris or Pat dies. We seek $\mathbb{E}(Z)$. In terms of random variables, since X and Y are the remaining lifetimes of Chris and Pat, respectively, Z is simply the minimum of X and Y. Thus, if $Z = \min\{X, Y\}$, then the distribution function of Z is

$$\begin{aligned} F_Z(z) = \mathbf{P}(Z \le z) = 1 - \mathbf{P}(Z > z) &= 1 - \mathbf{P}(X > z, Y > z) \\ &= 1 - \mathbf{P}(X > z)\mathbf{P}(Y > z) \\ &= 1 - [1 - F_X(z)][1 - F_Y(z)]. \end{aligned}$$

Since

$$F_X(x) = \begin{cases} 1 - e^{-x/15}, & x \ge 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad F_Y(y) = \begin{cases} 1 - e^{-y/30}, & y \ge 0, \\ 0, & y < 0, \end{cases}$$

and since $e^{-z/15}e^{-z/30} = e^{-z(1/15+1/30)} = e^{-z/10}$, we conclude that

$$F_Z(z) = \begin{cases} 1 - e^{-z/10}, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

Therefore, $f_Z(z) = \frac{1}{10}e^{-z/10}$, $z \geq 0$, implying that

$$\mathbb{E}(Z) = \int_{-\infty}^{\infty} z f_Z(z)\,\mathrm{d}z = \int_0^{\infty} \frac{1}{10} z e^{-z/10}\,\mathrm{d}z = 10 \int_0^{\infty} u e^{-u}\,\mathrm{d}u = 10 \cdot \Gamma(2) = 10.$$

2. (b) Note that Chris is the survivor if and only if $Y < X$. Therefore, by the law of total probability,

$$\mathbf{P}(Y < X) = \int_{-\infty}^{\infty} \mathbf{P}(Y < X | X = x) f_X(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathbf{P}(Y < x) f_X(x)\,\mathrm{d}x$$

where the last equality follows since X and Y are independent. Now

$$\mathbf{P}(Y < x) = \int_{-\infty}^{x} f_Y(y)\,\mathrm{d}y = \int_0^{x} \frac{1}{30} e^{-y/30}\,\mathrm{d}y = 1 - e^{-x/30}$$

and so

$$\begin{aligned} \mathbf{P}(Y < X) = \int_0^{\infty} [1 - e^{-x/30}] \cdot \frac{1}{15} e^{-x/15}\,\mathrm{d}x &= \frac{1}{15}\int_0^{\infty} e^{-x/15}\,\mathrm{d}x - \frac{1}{15}\int_0^{\infty} e^{-x/10}\,\mathrm{d}x \\ &= 1 - \frac{10}{15} \\ &= \frac{1}{3}. \end{aligned}$$

3. (a) If $X \sim \mathcal{N}(0, \sigma^2)$, then

$$\log[f_X(x)] = -\log[\sigma\sqrt{2\pi}] - \frac{x^2}{2\sigma^2}$$

and so

$$\begin{aligned} H(X) = -\int_{-\infty}^{\infty} f_X(x) \log[f_X(x)]\,\mathrm{d}x &= -\int_{-\infty}^{\infty} f_X(x) \left[-\log[\sigma\sqrt{2\pi}] - \frac{x^2}{2\sigma^2}\right] \mathrm{d}x \\ &= \log\left[\sigma\sqrt{2\pi}\right] \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x + \frac{1}{2\sigma^2}\int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x \\ &= \log\left[\sigma\sqrt{2\pi}\right] + \frac{1}{2\sigma^2}\mathbb{E}(X^2) \\ &= \log\left[\sigma\sqrt{2\pi}\right] + \frac{1}{2}. \end{aligned}$$

There are a number of equivalent ways to write $H(X)$. Here are some:

$$\log\left[\sigma\sqrt{2\pi}\right] + \frac{1}{2} = \log\left[\sigma\sqrt{2\pi}\right] + \frac{1}{2}\log[e] = \frac{1}{2}\log\left[2\pi\sigma^2\right] + \frac{1}{2}\log[e] = \frac{1}{2}\log\left[2\pi\sigma^2 e\right].$$

3. (b) If $Z = cY$, then the distribution function of Z is

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(cY \le z) = \mathbf{P}(Y \le z/c) = \int_{-\infty}^{z/c} f_Y(y)\,\mathrm{d}y$$

and so the density of Z is

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = f_Y(z/c) \cdot \frac{1}{c}.$$

Therefore, the entropy of Z is

$$H(Z) = -\int_{-\infty}^{\infty} f_Z(z) \log[f_Z(z)]\,\mathrm{d}z = -\int_{-\infty}^{\infty} \frac{1}{c} f_Y(z/c) \log\left[\frac{1}{c} f_Y(z/c)\right]\,\mathrm{d}z$$

Change variables by letting $y = z/c$ so that $\mathrm{d}y = (1/c)\,\mathrm{d}z$ implying

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{c} f_Y(z/c) \log\left[\frac{1}{c} f_Y(z/c)\right]\,\mathrm{d}z &= \int_{-\infty}^{\infty} f_Y(y) \log\left[\frac{1}{c} f_Y(y)\right]\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f_Y(y) \left(\log\left[\frac{1}{c}\right] + \log[f_Y(y)]\right)\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f_Y(y) \log\left[\frac{1}{c}\right]\,\mathrm{d}y + \int_{-\infty}^{\infty} f_Y(y) \log[f_Y(y)]\,\mathrm{d}y \\
&= \log\left[\frac{1}{c}\right] \int_{-\infty}^{\infty} f_Y(y)\,\mathrm{d}y + \int_{-\infty}^{\infty} f_Y(y) \log[f_Y(y)]\,\mathrm{d}y \\
&= -\log[c] - H(Y)
\end{aligned}$$

and so

$$H(X) = \log[c] + H(Y)$$

as required.

Note. The concept of entropy is extremely important in information theory. See

`http://en.wikipedia.org/wiki/Information_theory`

and

`http://en.wikipedia.org/wiki/Differential_entropy`

for further information.