Math/Stat 251 Midterm #2 – November 20, 2015

This exam has 3 problems and is worth a total of 50 points.

You have 50 minutes to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. ***Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.*** *Points will be deducted for incoherent, incorrect, and/or irrelevant statements. For questions with multiple parts, all parts are equally weighted.*

This exam is closed-book, although one $8\frac{1}{2} \times 11$ *double-sided page of handwritten notes is allowed. No other aids are allowed.*

You must answer all of the questions in the exam booklet provided.

Note that throughout this exam, all logarithms are taken to be natural logarithms so that $\log\theta = \ln\theta$*.*

1. (*24 points*) Suppose the distribution function of the continuous random variable X is

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ 1 - (1-x)^3, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

(a) Determine $f_X(x)$, the density function of X.

(b) Compute $\mathbb{E}(X)$, the expected value (or mean or average or expectation) of X.

(c) Compute $\mathbf{P}(-1 < X \le 1/2)$.

(d) Suppose that $Y = 1 - X$. Determine $F_Y(y)$, the distribution function of Y.

2. (*14 points*) A typical joint-first-to-die insurance policy held by two people pays out to the survivor upon the death of the other person. Suppose that Chris and Pat hold such an insurance policy. It is known that Chris' remaining lifetime (measured in years from today) is a random variable, denoted X, which has an exponential distribution with parameter 15 so that

$$f_X(x) = \frac{1}{15} e^{-x/15}, \quad x \ge 0.$$

It is also known that Pat's remaining lifetime (measured in years from today) is a random variable, denoted Y, which has an exponential distribution with parameter 30 so that

$$f_Y(y) = \frac{1}{30} e^{-y/30}, \quad y \ge 0.$$

You may assume that the remaining lifetimes of Chris and Pat are independent.

(a) Determine the expected time until the insurance policy pays out.

(b) Determine the probability that Chris is the survivor.

3. (*12 points*) Suppose that X is a continuous random variable with density $f_X(x)$. The *entropy* of X is defined as

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log[f_X(x)] \, \mathrm{d}x.$$

(a) Compute $H(X)$ if $X \sim \mathcal{N}(0, \sigma^2)$ so that the density of X is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty.$$

Hint. You may find it helpful to note that for such a random variable it was shown in class that $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) = \sigma^2$.

(b) If Y is a continuous random variable with density $f_Y(y)$, and the random variable Z is defined as $Z = cY$ where $c > 1$ is constant, carefully verify that

$$H(Z) = H(Y) + \log c.$$