Math/Stat 251 Fall 2015
Summary of Lecture from September 18, 2015

It is often the case that we need to compute probabilities associated with repeated trials. For example, consider the following simple model for the price of a stock. Each day there is a 75% chance that the value of the stock increases by \$1, while there is a 25% chance that the value of the stock decreases by \$1. If at time $t = 0$, the value of the stock is \$18, what is the probability that on day 2 (time $t = 2$), the value of the stock is \$20?

If S_2 denotes the stock's value on day 2, then $S_2 = 20$ if and only if the stock went up on both days. Intuitively, we find

$$\mathbf{P}(S_2 = 20) = (0.75)(0.75) = 0.5625.$$

The justification for this calculation lies in the fact that we are assuming each day's price movement is *independent* of any other day's movement. Thus, we might say that two events are independent provided that one does not influence the other. This is the correct intuition, but, unfortunately, we need to be more careful to make things mathematically precise. Perhaps a better way to remember independence is that *independence means multiply.*

Definition. Two events A and B are said to be *independent* if and only if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B).$$

Hence, we can formalize this stock price example as follows. Let A be the event that the stock price rises on the first day, and let B be the event that the stock price rises on the second day. The event that $S_2 = 20$ is the same as the event that the stock price rises on both days, namely $A \cap B$. Thus, since A and B are assumed to be independent, and since $\mathbf{P}(A) = \mathbf{P}(B) = 0.75$, we conclude that

$$\mathbf{P}(S_2 = 20) = \mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B) = (0.75)(0.75) = 0.5625$$

as before.

In fact, armed with this formal definition of independence, we can *justify* our solutions to problems #3 and #4 on the Thinking About Probability handout.

It is not always the case that two events under consideration are independent. In fact, very often, we want to consider dependent events. A standard example is when pharmaceutical companies conduct controlled clinical trials to test the efficacy of a new drug treatment being proposed. For simplicity, assume that patients are randomly assigned to one of two treatment regimes, say *old* or *new.* After letting the drug treatment run its course, patients are assessed to determine whether they show a marked *improvement* or *no improvement* in their condition. The pharmaceutical company desires that the new drug leads to an increase in patients with improved conditions. That is, they desire

$$\mathbf{P}(\text{improved condition given new drug}) > \mathbf{P}(\text{improved condition given old drug}).$$

If A is the event $A = \{\text{improved condition}\}$ and B is the event $B = \{\text{new drug}\}$, then the interest is in verifying that the *conditional probabilities* satisfy

$$\mathbf{P}(A \mid B) > \mathbf{P}(A \mid B^c).$$

In an introductory statistics class, you might have been presented with an exercise such as the following. There were 110 patients assigned at random to two different drug groups: 50 received the old drug and 60 received the new drug. In the new drug group 38 showed an improved condition, while in the old drug group 31 showed an improved condition. Assuming that there were no lurking/hidden variables and that improved condition can be attributable to the drug alone, do the data support the hypothesis that the new drug leads to an increase in improved conditions? To answer this question, you might have constructed a table such as

	new drug	old drug	totals
improvement	38	31	69
no improvement	22	19	41
totals	60	50	110

and performed a certain chi-squared test. Note that we will NOT be answering such a question in Math/Stat 251. However, this is a question that is most appropriately phrased in terms of conditional probabilities, and that IS a topic of Math/Stat 251. For instance, suppose that a single patient from among the 110 total patients is selected at random. Based on this data, we find

- $\mathbf{P}\left(\text{improvement} \mid \text{new drug}\right) = \frac{38}{60}$,
- $\mathbf{P}\left(\text{no improvement} \mid \text{new drug}\right) = \frac{22}{60}$,
- $\mathbf{P}\left(\text{improvement} \mid \text{old drug}\right) = \frac{31}{50}$,
- $\mathbf{P}\left(\text{no improvement} \mid \text{old drug}\right) = \frac{19}{50}$,
- $\mathbf{P}\left(\text{new drug} \mid \text{improvement}\right) = \frac{38}{69}$,
- $\mathbf{P}\left(\text{new drug} \mid \text{no improvement}\right) = \frac{22}{41}$,
- $\mathbf{P}\left(\text{old drug} \mid \text{improvement}\right) = \frac{31}{69}$,
- $\mathbf{P}\left(\text{old drug} \mid \text{no improvement}\right) = \frac{19}{41}$,

Informally, we see that conditional probabilities can be computed by looking at a *reduced sample space*; that is, by conditioning on an event, we restrict consideration to only those outcomes that belong to that event.

Definition. If A and B are events, then the *conditional probability of A given B* is defined as

$$\mathbf{P}\left(A|B\right) = \frac{\mathbf{P}\left(A \cap B\right)}{\mathbf{P}\left(B\right)}$$

provided that $\mathbf{P}\left(B\right) > 0$.