Math/Stat 251 Fall 2015
Summary of Lecture from September 14, 2015

Our goal is to devise a mathematical model of a chance experiment; that is, an experiment whose outcome is randomly one of many possibilities. A priori, we do not know which outcome will result.

Formally, the *sample space* denoted by S consists of all possible *outcomes* of the experiment. Since we do not know which outcome will result, we assign probabilities to the various outcomes in a *reasonable* way to indicate the relative likelihood with which we believe those outcomes to occur. The actual assignment of probabilities in a given situation might be a philosophical matter. However, we will not engage in that discussion here. We will assume that there is a probability assignment.

In addition to assigning probabilities to individual outcomes, we might want to assign probabilities to more sophisticated collections of outcomes. These are known as events. Thus, an *event* is a subset of the sample space (to use the language of set theory); that is, an event is simply a collection of outcomes.

As we have already noted, a probability is, intuitively, a measure of the relative likelihood of the occurrence of an event. Therefore, several reasonable properties that we might expect a probability to satisfy are the following.

(i) The probability that *something* happens should be 1; that is, $\mathbf{P}(S) = 1$.

(ii) The probability that *nothing* happens should be 0; that is, $\mathbf{P}(\emptyset) = 0$.

(iii) The probability of any event should be between 0 and 1; that is, if A is an event, then $0 \leq \mathbf{P}(A) \leq 1$.

(iv) The probability of an event and its complement should sum to 1; that is, if A is an event, then the probability of A^c (the complement of A in S) should be $\mathbf{P}(A^c) = 1 - \mathbf{P}(A)$.

(v) The probability of a disjoint union of events should be the sum of the individual probabilities; that is, if A and B are disjoint events (meaning $A \cap B = \emptyset$), then the probability of either A or B happening (meaning $A \cup B$) should be $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$.

Hence, we see that a probability $\mathbf{P}$ should be defined as a function from a collection of events to the real numbers in $[0, 1]$.

In addition to considering the probabilities of events, we might want to *bet* on a particular outcome. Although the wager might be fixed, the payout of the bet depends on the outcome observed. We describe such a bet as a *random variable*. Formally, a random variable is defined as a function from the sample space S to the real numbers $\mathbb{R}$. The letter X is traditionally used to denote a random variable so that we might write $X : S \to \mathbb{R}$.

Example. Suppose that we wish to model the experiment of flipping a fair coin twice. If we take our sample space to be

$$S = \{HH, HT, TH, TT\},$$

then, since it is assumed that the coin is fair, it is reasonable to declare that

$$\mathbf{P}(\{HH\}) = \mathbf{P}(\{HT\}) = \mathbf{P}(\{TH\}) = \mathbf{P}(\{TT\}) = \frac{1}{4}.$$

Using the probabilities for the individual outcomes along with **(v)** we can determine probabilities for various events. For instance, if A is the event that at least one head appears, then $A = \{HH, HT, TH\}$ and so

$$\begin{aligned}\mathbf{P}(A) = \mathbf{P}(\{HH, HT, TH\}) = \mathbf{P}(\{HH\} \text{ or } \{HT\} \text{ or } \{TH\}) &= \mathbf{P}(\{HH\}) + \mathbf{P}(\{HT\}) + \mathbf{P}(\{TH\}) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ &= \frac{3}{4}.\end{aligned}$$

If we let B be the event that no head appears, then $B = A^c$ and so using **(iv)** we find

$$\mathbf{P}(B) = \mathbf{P}(A^c) = 1 - \mathbf{P}(A) = 1 - \frac{3}{4} = \frac{1}{4}.$$

Of course, we could have solved this easy problem directly by noting that

$$\mathbf{P}(B) = \mathbf{P}(TT) = \frac{1}{4},$$

but the point was to illustrate **(iv)** in a simple setting. Suppose that you and I make the following bet. You pay me \$3, you flip the coin twice, and I pay you \$4 for every head that appears. If X denotes your net winnings, then X is a random variable whose value depends on the outcome observed. That is, $X : S \to \mathbb{R}$ is given explicitly by

$$\begin{aligned}X(HH) &= 2 \times 4 - 3 = 5, \\ X(TH) &= 1 \times 4 - 3 = 1, \\ X(HT) &= 1 \times 4 - 3 = 1, \\ X(TT) &= 0 \times 4 - 3 = -3.\end{aligned}$$

Notice that we can use the probabilities for the individual outcomes to determine probabilities for the values of the random variable. That is,

$$\begin{aligned}&\mathbf{P}(X = 5) = \mathbf{P}(\{HH\}) = \frac{1}{4}, \\ &\mathbf{P}(X = 1) = \mathbf{P}(\{HT\} \text{ or } \{TH\}) = \mathbf{P}(\{HT\}) + \mathbf{P}(\{TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \\ &\mathbf{P}(X = -3) = \mathbf{P}(\{TT\}) = \frac{1}{4}.\end{aligned}$$