Math/Stat 251 Fall 2015
More on the Poisson Process (November 30, 2015)

A Poisson process is a model of arrivals. The two assumptions are that

- the waiting times between arrivals are independent and identically distributed $\text{Exp}(\lambda)$ random variables, and
- the number of arrivals by a given time is a $\text{Poisson}(\lambda t)$ random variable.

We sometimes call λ the *rate* or the *intensity* of the Poisson process.

Suppose that X_t denotes the number of arrivals by time t. The second assumption states that $X_t \sim \text{Poisson}(\lambda t)$. That is,

$$\mathbf{P}(X_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

for any $k = 0, 1, 2, \ldots$.

Recall that the exponential distribution is the unique continuous distribution to have the memoryless property. This means that we can reset a Poisson process at any given time. Symbolically, if s and t are two times with $s < t$, then whatever happens between times s and t is independent of what happened before time s and only depends on the amount of time between s and t; that is, if $s < t$ and $j < k$, then

$$\mathbf{P}(X_t = k \mid X_s = j) = \mathbf{P}(X_{t-s} = k - j).$$

Example. If we want the probability that there are 4 arrivals by time 7 given that there are 3 arrivals by time 5, the only way for this to happen is if there is exactly 1 arrival between times 5 and 7. By resetting the Poisson process at time 5, we must have 1 arrival in the next 2 units of time. That is,

$$\mathbf{P}(X_7 = 4 \mid X_5 = 3) = \mathbf{P}(X_2 = 1).$$

It is often the case that we need to combine the previous two ideas with the definition of conditional probability.

Example. If we want the probability that there are 3 arrivals by time 5 given that there are 4 arrivals by time 7, then we want to compute $\mathbf{P}(X_5 = 3 \mid X_7 = 4)$. However, we must use the definition of conditional probability to turn this into the form above. That is,

$$\mathbf{P}(X_5 = 3 \mid X_7 = 4) = \frac{\mathbf{P}(X_7 = 4 \mid X_5 = 3)\,\mathbf{P}(X_5 = 3)}{\mathbf{P}(X_7 = 4)} = \frac{\mathbf{P}(X_2 = 1)\,\mathbf{P}(X_5 = 3)}{\mathbf{P}(X_7 = 4)}$$

We know that $X_t \sim \text{Poisson}(\lambda t)$ from the second assumption above so that $X_2 \sim \text{Poisson}(2\lambda)$, $X_5 \sim \text{Poisson}(5\lambda)$, and $X_7 \sim \text{Poisson}(7\lambda)$. Hence,

$$\frac{\mathbf{P}(X_2 = 1)\,\mathbf{P}(X_5 = 3)}{\mathbf{P}(X_7 = 4)} = \frac{\frac{(2\lambda)^1}{1!}e^{-2\lambda} \cdot \frac{(5\lambda)^3}{3!}e^{-5\lambda}}{\frac{(7\lambda)^4}{4!}e^{-7\lambda}} = \frac{\frac{(2\lambda)^1}{1!}\frac{(5\lambda)^3}{3!}}{\frac{(7\lambda)^4}{4!}} = \frac{4! \cdot 2^1 \cdot 5^3}{1! \cdot 3! \cdot 7^4}.$$

Example. On any given day, the number of cigarettes that Keith Richards has lit since he woke up follows a Poisson process with an intensity of $\lambda = 4$ cigarettes per hour. You may assume that Keith Richards wakes up at 10:00 a.m. every day.

(a) What is the expected time of the day at which he lights his 8th cigarette?

(b) What is the probability that he lights 3 cigarettes or more between noon and 1 p.m.?

Example. In New York City, subway trains are notoriously unreliable. In fact, subway trains arrive at Grand Central Station according to a Poisson process with a rate (or intensity) of 1 train every 4 minutes.

(a) How many subway trains are expected to arrive in one hour? (Recall that there are 60 minutes in one hour.)

(b) Suppose that Christian is going to work and arrives at Grand Central Station at 8:00 a.m. just as a subway train is departing. What is the probability that Christian will wait at least 8 minutes for the next train to arrive?

(c) Suppose that, after work, Christian and Veronica have agreed to meet at Grand Central Station at 5:00 p.m. Christian is punctual and arrives at 5:00 p.m. However, Veronica is late leaving work and so she arrives at Grand Central Station at 5:16 p.m. What is the probability that at least 3 subway trains pass Christian while he is waiting for Veronica?