Math/Stat 251 Fall 2015
Some Properties of Expected Value

> *Individual behaviour may be erratic, but aggregate behaviour is often quite predictable.*

Example. Suppose that $X \sim \text{Exp}(\lambda)$. Compute $m(t)$, the moment generating function of X, and use this to compute $\mathbb{E}(X^k)$.

Solution. Since $X \sim \text{Exp}(\lambda)$, we know that $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Therefore,

$$m(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,\mathrm{d}x = \lambda \int_0^{\infty} e^{tx} e^{-\lambda x}\,\mathrm{d}x = \lambda \int_0^{\infty} e^{-(\lambda - t)x}\,\mathrm{d}x = \frac{\lambda}{\lambda - t}$$

and so

$$m^{(k)}(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\frac{\lambda}{\lambda - t}\right) = \frac{k!\lambda}{(\lambda - t)^{k+1}}.$$

Thus,

$$\mathbb{E}(X^k) = m^{(k)}(0) = \frac{k!\lambda}{(\lambda - 0)^{k+1}} = \frac{k!\lambda}{\lambda^{k+1}} = \frac{k!}{\lambda^k}.$$

In fact, we can verify our answer by noting that $\mathbb{E}(X^k)$ could have been computed directly. That is,

$$\mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k f(x)\,\mathrm{d}x = \lambda \int_{-\infty}^{\infty} x^k e^{-\lambda x}\,\mathrm{d}x = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}$$

using properties of the Gamma function.

We already know that the first moment represents the mean or average value of the distribution. The mean should not be confused with the median which is that value of a for which $\mathbf{P}(X \leq a) = 0.5$. In other words, half the area under the density curve is less than the median and half the area under the density curve is greater than the median. If the density curve is symmetric, then the mean and the median are equal.

Example. Suppose that $X \sim \text{Exp}(\lambda)$. The median of X is that value a satisfying

$$0.5 = \mathbf{P}(X \leq a) = \int_0^a \lambda e^{-\lambda x}\,\mathrm{d}x = 1 - e^{-\lambda a}.$$

That is,

$$a = \frac{\log 2}{\lambda}.$$

Notice that the median of X does not equal the mean $\mathbb{E}(X) = \lambda^{-1}$. This is not surprising since the density curve for an exponential random variable is asymmetric.

Suppose that X is a continuous random variable with density f. If we interpret X as the payout of a bet of a chance experiment, then we might be interested in making a sidebet on that experiment; in other words, a bet on the bet. We can interpret this sidebet as $g(X)$, which is itself a random variable. Another scenario might be that X represents an observable of a physical system subjected to random disturbances. We may then want to represent X in different units. For instance, suppose that X is the temperature measured in degrees celsius. If we want to express the temperature in degrees fahrenheit, then we need to consider

$$g(X) = \frac{9}{5}X + 32.$$

In general, we define the expected value of $g(X)$ as

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,\mathrm{d}x.$$

This is sometimes called the *Law of the Unconscious Statistician*!

Expectation is Linear

If $g(X) = aX + b$ where a and b are constants, then

$$\mathbb{E}(aX+b) = \int_{-\infty}^{\infty} (ax+b)f(x)\,\mathrm{d}x = a\int_{-\infty}^{\infty} xf(x)\,\mathrm{d}x + b\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = a\mathbb{E}(X) + b.$$

Example. The average temperature of a Tim Horton's cup of coffee is 90°C. What is the average temperature of such a cup of coffee in degrees fahrenheit?

Solution. We know that if X represents the temperature of a randomly selected cup of Tim Horton's coffee, then $\mathbb{E}(X) = 90$. Thus, the average temperature in degrees fahrenheit is

$$\mathbb{E}\left(\frac{9}{5}X + 32\right) = \frac{9}{5}\mathbb{E}(X) + 32 = \frac{9}{5}(90) + 32 = 194.$$

Expectation is Additive

If X and Y are random variables, then

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

The proof will be delayed until we have studied *joint distributions*. Basically this says that the average of averages is the same thing as the overall average.

Exercise. Based on studies from previous Hallowe'ens, I have determined that trick-or-treating ghouls arrive at my door as follows. The ghouls arrive independently and the time in minutes between the arrival of successive ghouls is exponentially distributed with parameter $\lambda = 2$. If I leave my jack-o-latern lit for two hours (signifying that the ghouls are welcome to ring my bell) and my pumpkin contains 100 candy bars, do I expect to have enough candy bars to give one to every ghoul who trick-or-treats?