Math/Stat 251 Fall 2015
Some Examples of a One-Dimensional Change of Variables (November 2, 2015)

Suppose that X is a continuous random variable and that $Y = g(X)$ for some continuous function $g : \mathbb{R} \to \mathbb{R}$ so that Y is itself a continuous random variable. It is often the case in practice that one knows the density function of X and seeks the density function of Y. Fortunately, if g is a nice function (as it usually is in practice), then it is straightforward to determine the density of Y from first principles. Basically, one starts with the definition of the distribution function of Y substitutes in $Y = g(X)$, and solves for X. This produces an integral expression involving the density function of X which can then be differentiated using the fundamental theorem of calculus to yield the density function for Y. Sometimes this is called a *one-dimensional change of variables*. The following examples illustrate this technique. Remember that in order to use the fundamental theorem of calculus, it must be the case that a variable appears in the upper limit of integration and that no variable appears in the lower limit of integration.

Example. Suppose that $X \sim \mathcal{N}(0,1)$. Let $Y = e^X$. Determine the density function of Y.

Solution. Let $Y = e^X$. For $y > 0$, the distribution function of Y is

$$F_Y(y) = \mathbf{P}\left(Y \le y\right) = \mathbf{P}\left(e^X \le y\right) = \mathbf{P}\left(X \le \log y\right) = \int_{-\infty}^{\log y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x$$

so that the density function of Y is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{\log y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} e^{-(\log y)^2/2} \cdot \frac{\mathrm{d}}{\mathrm{d}y}\log y = \frac{1}{y\sqrt{2\pi}} e^{-(\log y)^2/2}$$

for $y > 0$. *The random variable Y is an example of a log-normal random variable which is regularly encountered in the mathematical theory of stock option pricing.*

Example. Suppose that $X \sim \mathcal{N}(0,1)$. Let $Y = X^2$. Determine the density function of Y.

Solution. Let $Y = X^2$ so that

$$F_Y(y) = \mathbf{P}\left(Y \le y\right) = \mathbf{P}\left(X^2 \le y\right).$$

Note that since X can take on *any* real value, we have

$$\begin{aligned}
\mathbf{P}\left(X^2 \le y\right) = \mathbf{P}\left(-\sqrt{y} \le X \le \sqrt{y}\right) &= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,\mathrm{d}x \\
&= \int_{-\sqrt{y}}^{0} f_X(x)\,\mathrm{d}x + \int_{0}^{\sqrt{y}} f_X(x)\,\mathrm{d}x \\
&= \int_{0}^{\sqrt{y}} f_X(x)\,\mathrm{d}x - \int_{0}^{-\sqrt{y}} f_X(x)\,\mathrm{d}x \\
&= \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x - \int_{0}^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,\mathrm{d}x,
\end{aligned}$$

and so

$$\begin{aligned} f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}y}\int_0^{-\sqrt{y}} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}e^{-(\sqrt{y})^2/2}\cdot\frac{\mathrm{d}}{\mathrm{d}y}(\sqrt{y}) - \frac{1}{\sqrt{2\pi}}e^{-(-\sqrt{y})^2/2}\cdot\frac{\mathrm{d}}{\mathrm{d}y}(-\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi}}e^{-y/2}\cdot\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{-y/2}\cdot\frac{1}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}e^{-y/2} \end{aligned}$$

for $y > 0$. *Note that the random variable Y has a* Gamma$(1/2, 1/2)$ *distribution, or equivalently, $Y \sim \chi^2(1)$ and often appears in statistical inference. It also sometimes appears in the mathematical theory of stock option pricing.*

Example. Suppose that $X \in \Gamma(a, b)$ so that the density of X is

$$f_X(x) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}$$

for $x \geq 0$. Let $Y = 1/X$. Determine the density function of Y.

Solution. Let $Y = 1/X$. For $y > 0$, the distribution function of Y is

$$\begin{aligned} F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(1/X \leq y) = \mathbf{P}(X \geq 1/y) &= 1 - \mathbf{P}(X < 1/y) \\ &= 1 - \int_{-\infty}^{1/y} \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}\,\mathrm{d}x \end{aligned}$$

so that the density function of Y is

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}F_y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - \int_{-\infty}^{1/y} \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}\,\mathrm{d}x\right) = \frac{b^a}{\Gamma(a)}y^{1-a}e^{-b/y}\cdot\frac{1}{y^2} = \frac{b^a}{\Gamma(a)}y^{-a-1}e^{-b/y}$$

for $y > 0$. *The random variable Y is an example of an inverse gamma random variable with parameters a and b and is used primarily in Bayesian statistics though it sometimes finds applications in actuarial science.*